

# MODERATE DEVIATIONS PRINCIPLE FOR EMPIRICAL COVARIANCE FROM A UNIT ROOT

YU MIAO, YAN-LING WANG, AND GUANG-YU YANG

ABSTRACT. In the present paper, we consider the linear autoregressive model in  $\mathbb{R}$ ,

$$X_{k,n} = \theta_n X_{k,n-1} + \xi_k, \quad k = 0, 1, \dots, n, \quad n \geq 1$$

where  $\theta_n \in [0, 1)$  is unknown,  $(\xi_k)_{k \in \mathbb{Z}}$  is a sequence of centered i.i.d. r.v. valued in  $\mathbb{R}$  representing the noise. When  $\theta_n \rightarrow 1$ , the moderate deviations principle for empirical covariance is discussed and as statistical applications we provide the moderate deviation estimates of the least square and the Yule-Walker estimators of the parameter  $\theta_n$ .

## 1. INTRODUCTION

There is a great deal of the econometric literature of the last 20 years which has focused on the issue of testing for the unit root hypothesis in economic time series. Regression asymptotics with roots at or near unity have played an important role in time series econometrics. This has been typically done by using autoregressive models with fixed coefficients and then testing for the autoregressive parameter being equal to 1 [5, 6]. More recently, some attention has been dedicated to random coefficient autoregressive models. This way of handling the data allows for large shocks in the dynamic structure of the model, and also for some flexibility in the features of the volatility of the series, which are not available in fixed coefficient autoregressive models.

In the present paper, we consider the following linear autoregressive model in  $\mathbb{R}$ ,

$$X_{k,n} = \theta_n X_{k-1,n} + \xi_k, \quad k = 0, 1, \dots, n, \quad n \geq 1 \quad (1.1)$$

where  $\theta_n \in \Theta \subset \mathbb{R}$  (the space of parameter) is unknown,  $(\xi_k)_{k \in \mathbb{Z}}$  is a sequence of centered i.i.d. r.v. valued in  $\mathbb{R}$  representing the noise and which is independent of  $X_{0,n}$ , and  $(X_{k,n})_{0 \leq k \leq n}$  is observed. For every  $n \geq 0$ , assume that the law of  $X_{0,n}$  is invariant (or equivalently  $(X_{k,n})_{0 \leq k \leq n}$  is stationary), it is easy to see a stationary solution to (1.1), which is given by

$$X_{k,n} = \sum_{p=0}^{\infty} \theta_n^p \xi_{k-p}, \quad k \geq 0$$

only if  $|\theta_n| < 1$ .

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2000 *Mathematics Subject Classification.* 60F10, 60G10, 62J05.

*Key words and phrases.* Moderate deviations principle, empirical covariance unit root, autoregressive model.

It is not difficult to see that the linear autoregressive model (1.1) is a special moving average process. A general moving average process is given by

$$X_n := \sum_{j=-\infty}^{+\infty} a_{j-n} \xi_j = \sum_{j=-\infty}^{+\infty} a_j \xi_{n+j}, \quad \forall n \in \mathbb{Z},$$

where  $(\xi_n)_{n \in \mathbb{Z}}$  is i.i.d.,  $(a_n)_{n \in \mathbb{Z}}$  is a sequence of real numbers such that

$$\sum_{n \in \mathbb{Z}} |a_n|^2 < \infty.$$

There are two important issues for the model (1.1): (1) the estimate of the covariance  $Cov(X_{0,n}, X_{l,n}) := \mathbb{E}(X_{0,n} - \mathbb{E}X_{0,n})(X_{l,n} - \mathbb{E}X_{l,n})$ ; (2) the estimate of  $\theta_n$ . The most natural estimator of  $Cov(X_{0,n}, X_{l,n})$  ( $l \geq 0$ ) is given by the empirical covariance (with the given sample  $(X_{k,n})_{0 \leq k \leq n-l}$ )

$$C_{l,n}^* = \frac{1}{n-l} \sum_{k=1}^{n-l} X_{k+l,n} X_{k,n} \quad (1.2)$$

and for estimating  $\theta_n$ , the following two estimators are widely used:

(i) Least Square Estimator:

$$\hat{\theta}_n = \frac{\sum_{k=1}^n X_{k,n} X_{k-1,n}}{\sum_{k=1}^n X_{k-1,n}^2}. \quad (1.3)$$

(ii) Yule-Walker Estimator:

$$\tilde{\theta}_n = \frac{\sum_{k=1}^n X_{k,n} X_{k-1,n}}{\sum_{k=0}^n X_{k,n}^2}. \quad (1.4)$$

In this paper, we are concerned with the moderate deviations principle of the covariance estimation  $C_{l,n}^*$  and the parameter estimators  $\hat{\theta}_n, \tilde{\theta}_n$  for the linear autoregressive model under the case:  $\theta_n \in [0, 1)$  and  $\theta_n \rightarrow 1$ .

The study on large deviations and moderate deviation are relatively recent and these works concentrate almost on the case of the fixed autoregressive coefficient  $\theta_n \equiv \theta \in (-1, 1)$ , i.e.,

$$X_n = \theta X_{n-1} + \xi_n, \quad n \geq 0. \quad (1.5)$$

For the Gaussian case (i.e., the noise  $\xi$  is assumed Gaussian), this subject is opened by Donsker and Varadhan [8] who proved the level-3 large deviation principle (the definition of large deviations of level-3 could be found in [9]) for general stationary Gaussian processes under the continuity of the spectral function. Bryc and Dembo [1] proved for the first the large and moderate deviation principles for the empirical variance  $C_{0,n}^* (= n^{-1} \sum_{k=1}^n X_k^2)$  even for general stationary Gaussian processes. Bercu et al. [2] proved the large deviation principle for  $C_{l,n}^* (= n^{-1} \sum_{k=1}^n X_{k+l} X_k)$ ,  $l \geq 0$  (which is much more delicate than  $C_{0,n}^*$ ) and for  $\hat{\theta}_n, \tilde{\theta}_n$ .

For the Non-Gaussian case, Wu [17] first extended Donsker-Varadhan's theorem on large deviations of level-3 from stationary Gaussian processes to general moving average processes under the Gaussian integrability condition on the driven variable  $\xi$ . Djellout et al. [7] established, in the one-dimensional case, moderate deviation

principle for non-linear functionals of general moving average processes covering the case of  $C_{l,n}^*$  and for the periodogram, but under the assumption that the law of the driven random variable  $\xi$  satisfies the log-Sobolev inequality, stronger than the Gaussian integrability in [17].

For the case of Hilbertian autoregressive model with driven random variable  $\xi$  satisfying the Gaussian integrability condition, in which  $\{\xi_k, X_k\}_{k \in \mathbb{Z}}$  take values in some separable Hilbert space  $H$ , Mas and Menneteau [11] established large and moderate deviation for the empirical mean  $\bar{X}_n = \frac{1}{n} \sum_{k=1}^n X_k$ , and moderate deviation for the empirical variance matrix  $\frac{1}{n} \sum_{k=1}^n X_k \otimes X_k$ , where  $x \otimes y$  ( $x, y \in H$ ) denotes the linear operator from  $H$  to  $H$ ,

$$x \otimes y : h \in H \rightarrow \langle x, h \rangle y,$$

extending the result of Bryc-Dembo [1] from  $\mathbb{R}^d$  to  $H$ , and especially from Gaussian case to general sub-Gaussian case. Furthermore, Menneteau [12] obtained some laws of the iterated logarithm in Hilbertian autoregressive models for the empirical covariance  $\frac{1}{n} \sum_{k=1}^n X_k \otimes X_k$ . Recently, Miao and Shen [14] obtained a moderate deviations principle for  $C_{n,l}^*$  of the autoregressive process (1.5), which removed the assumption of log-Sobolev inequality on the driven variable in [7], for the particular but important auto-regression model. In addition, they provided the moderate deviation estimates of the least squares and the Yule-Walker estimators of the unknown parameter of an autoregressive process. In [15], the author also considered the discounted large deviation principle for the autoregressive processes (1.5).

Our main purpose in the paper is to extend the moderate deviations principle for the empirical covariance from the case  $\theta_n = \theta$  to the case  $\theta_n \rightarrow 1$ . The method of proof relies mainly on a moderate deviation for triangular arrays of finitely-dependent sequences and the exponential approximation. This paper is organized as follows. The next section is devoted to the descriptions of our main results and their statistical applications. In Section 3, we give some preparations and develop a new moderate deviation for  $m$ -dependent sequence with unbounded  $m$ . The proofs of main results are obtained in the remaining sections.

## 2. MAIN RESULTS

**2.1. Assumptions.** Let  $\{\xi_n\}_{n \in \mathbb{Z}}$  be a sequence of real valued centered i.i.d. random variables, and suppose that the following conditions hold:

- (1) the unknown parameter  $\theta_n$  satisfies  $\theta_n \in [0, 1)$ ,  $\theta_n \rightarrow 1$ ;
- (2)  $\mathbb{E}\xi_0 = 0$  and  $\xi_0$  satisfies the Gaussian integrability condition, i.e., there exists  $\alpha > 0$ , such that

$$\mathbb{E}e^{\alpha \xi_0^2} < \infty;$$

- (3) the moderate deviation scale  $(b_n)$  is a sequence of positive numbers satisfying

$$b_n \rightarrow \infty, \quad \frac{\sqrt{n}(1 - \theta_n)^2}{b_n} \rightarrow \infty.$$

Here we need to note that the condition (3) implies

$$\lim_{n \rightarrow \infty} n(1 - \theta_n) = \infty, \quad \text{and} \quad \frac{\sqrt{n}}{b_n} \rightarrow \infty.$$

**2.2. Moderate deviations principle.** The following is our main theorem.

**Theorem 2.1.** *Assume that the conditions (1), (2) and (3) are satisfied and let  $M$  be a non-negative integer, then for all  $r > 0$ , when  $0 \leq l \leq M$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{(1 - \theta_n^2)^{3/2} \sqrt{n}}{b_n} |C_{l,n}^* - \mathbb{E}C_{l,n}^*| \geq r \right) = -\frac{r^2}{8(\mathbb{E}\xi_0^2)^2}. \quad (2.1)$$

**Remark 2.1.** Since  $M$  is fixed and  $l$  is finite, then the form (2.1) is equivalent to

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{(1 - \theta_n^2)^{3/2} \sqrt{n-l}}{b_n} |C_{l,n}^* - \mathbb{E}C_{l,n}^*| \geq r \right) = -\frac{r^2}{8(\mathbb{E}\xi_0^2)^2}. \quad (2.2)$$

In the process of proving Theorem 2.1, we often use the form (2.2) in order to avoid extra explanation.

The following result supplies a moderate deviation for the linear combination of the empirical covariance. For the case that the unknown parameter  $\theta_n$  is fixed ( $\theta_n = \theta$ ), we can succeed in obtaining the moderate deviation of the parameter estimators  $\hat{\theta}_n, \tilde{\theta}_n$  by utilizing the following result directly.

**Theorem 2.2.** *Under the conditions in Theorem 2.1, for any  $n$  and  $0 \leq l \leq M$ , let  $\{a_{l,n}\}$  be a sequences of real numbers with  $\lim_{n \rightarrow \infty} a_{l,n} = a_l$ , and assume that  $a_l \neq 0$  for some  $0 \leq l \leq M$ . Then for any  $r > 0$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{(1 - \theta_n^2)^{3/2} \sqrt{n}}{b_n} \left| \sum_{l=0}^M a_{l,n} (C_{l,n}^* - \mathbb{E}C_{l,n}^*) \right| \geq r \right) = -\frac{r^2}{2\Sigma^2}$$

where

$$\Sigma^2 = 4 \left( \sum_{j=0}^M a_j \right)^2 (\mathbb{E}\xi_0^2)^2.$$

**Remark 2.2.** Under the conditions of Theorem 2.2, if  $\sum_{j=0}^M a_j = 1$ , then we have

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{(1 - \theta_n^2)^{3/2} \sqrt{n}}{b_n} \left| \sum_{l=0}^M a_{l,n} (C_{l,n}^* - \mathbb{E}C_{l,n}^*) \right| \geq r \right) = -\frac{r^2}{8(\mathbb{E}\xi_0^2)^2}.$$

In particular, Theorem 2.1 holds, if there exists some  $0 < l < M$ , such that

$$a_k = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}.$$

**2.3. Applications.** In the subsection, we provide a statistical application. More precisely, we shall apply the method of proving Theorem 2.1 and 2.2 to the least squares estimator  $\hat{\theta}_n^2$  and the Yule-Walker estimator  $\tilde{\theta}_n$ .

**Proposition 2.1.** *Assume that the conditions (1), (2) and (3) are satisfied, then for any  $r > 0$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{\sqrt{n}}{b_n(1 - \theta_n^2)^{1/2}} |\hat{\theta}_n - \theta_n| \geq r \right) = -\frac{r^2}{2}$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{\sqrt{n}}{b_n(1 - \theta_n^2)^{1/2}} |\tilde{\theta}_n - \theta_n| \geq r \right) = -\frac{r^2}{2}.$$

**Remark 2.3.** A recent paper by Giraitis and Phillips [10] (also see, Phillips and Magdalinos [16]), established the asymptotic distribution of the least square estimator  $\hat{\theta}_n$  in a stationary first-order AR model when  $n(1 - \theta_n) \rightarrow \infty$ , i.e.,

$$(1 - \theta_n^2)^{-1/2} n^{1/2} (\hat{\theta}_n - \theta_n) \xrightarrow{d} N(0, 1).$$

**Remark 2.4.** For the case of  $\theta_n \equiv \theta \in (-1, 1)$ , Djellout et al. [7] derived the moderate deviations of  $\hat{\theta}_n$  and  $\tilde{\theta}_n$  as a consequence of their general results on the moderate deviation of moving average processes, but with an extra and strong condition that the law of  $\xi_0$  satisfies a log-Sobolev inequality (though their method go far beyond the regression model). In [14], the authors gave the moderate deviations of  $\hat{\theta}_n$  and  $\tilde{\theta}_n$ , where they removed the assumption of log-Sobolev inequality on the driven variable.

### 3. SOME PREPARATIONS AND AUXILIARY RESULTS

**3.1. Autoregressive representation for the covariance process.** For any  $n$ , by the stationarity of  $X_{k,n}$  ( $k = 0, 1, \dots, n$ ), the distribution law of  $X_{k+l,n}X_{k,n}$  is the same with  $X_{l,n}X_{0,n}$ . For any  $0 \leq l \leq M$ , let  $C_{l,n} := \mathbb{E}X_{k+l,n}X_{k,n}$  and it is easy to check that

$$C_{l,n} = \theta_n^l \mathbb{E}X_{0,n}^2 = \theta_n^l \sum_{k=0}^{\infty} \theta_n^{2k} \mathbb{E}\xi_0^2 = \mathbb{E}C_{l,n}^* \quad (3.1)$$

where  $C_{l,n}^*$  is defined in (1.2). In addition, let

$$Z_{k,l,n} = X_{k+l,n}X_{k,n} - C_{l,n}, \quad U_{k,l,n} = \theta_n X_{k+l-1,n}\xi_k + \theta_n \xi_{k+l}X_{k-1,n} + \xi_{k+l}\xi_k - \theta_n^l \mathbb{E}\xi_0^2. \quad (3.2)$$

We have the following autoregressive representation for the covariance process.

**Lemma 3.1.** *Under the above notions, for any  $n > l$ , we have*

$$Z_{k,l,n} = \theta_n^2 Z_{k-1,l,n} + U_{k,l,n}, \quad k = 1, \dots, n-l, \quad (3.3)$$

and

$$C_{l,n}^* - C_{l,n} = \frac{\bar{U}_{l,n}}{(1 - \theta_n^2)} + \frac{\theta_n^2 (Z_{0,l,n} - Z_{n-l,l,n})}{(n-l)(1 - \theta_n^2)}, \quad (3.4)$$

where

$$\bar{U}_{l,n} = \frac{1}{n-l} \sum_{k=1}^{n-l} U_{k,l,n}.$$

*Proof.* The proof of the lemma is easy, so omitted.  $\square$

**Lemma 3.2.** *Under the assumptions of Theorem 2.1, for any  $0 \leq l \leq M$  and  $r > 0$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{\sqrt{1 - \theta_n^2} |Z_{0,l,n} - Z_{n-l,l,n}|}{b_n \sqrt{n-l}} \geq r \right) = -\infty.$$

*Proof.* For every  $n$ , from the stationarity of  $\{X_{k,n}\}_{0 \leq k \leq n}$ , we have

$$\begin{aligned}
& \mathbb{P} \left( |Z_{0,l,n} - Z_{n-l,l,n}| \geq \frac{rb_n \sqrt{n-l}}{\sqrt{1-\theta_n^2}} \right) \\
&= \mathbb{P} \left( |X_{l,n}X_{0,n} - X_{n,n}X_{n-l,n}| \geq \frac{rb_n \sqrt{n-l}}{\sqrt{1-\theta_n^2}} \right) \\
&\leq 2\mathbb{P} \left( |X_{0,n}X_{l,n}| \geq \frac{rb_n \sqrt{n-l}}{2\sqrt{1-\theta_n^2}} \right) \\
&\leq 4\mathbb{P} \left( X_{0,n}^2 \geq \frac{rb_n \sqrt{n-l}}{2\sqrt{1-\theta_n^2}} \right),
\end{aligned}$$

where the last inequality follows from the well-known:

$$2|X_{0,n}X_{l,n}| \leq X_{0,n}^2 + X_{l,n}^2.$$

Now since

$$X_{0,n}^2 = \left( \sum_{p=0}^{\infty} \theta_n^p \xi_{-p} \right)^2 \leq \left( \sum_{p=0}^{\infty} \theta_n^p \right) \left( \sum_{p=0}^{\infty} \theta_n^p \xi_{-p}^2 \right) = \frac{1}{1-\theta_n} \sum_{p=0}^{\infty} \theta_n^p \xi_{-p}^2$$

and Markov's inequality, we have for  $\lambda_n := (1-\theta_n)^2 \alpha$ ,

$$\mathbb{P} \left( X_{0,n}^2 \geq \frac{rb_n \sqrt{n-l}}{2\sqrt{1-\theta_n^2}} \right) \leq \exp \left( -\lambda_n r \frac{b_n \sqrt{n-l}}{2\sqrt{1-\theta_n^2}} \right) \mathbb{E} e^{\lambda_n X_{0,n}^2}.$$

But by Jensen's inequality,

$$\mathbb{E} e^{\lambda_n X_{0,n}^2} \leq \mathbb{E} \exp \left( (1-\theta_n) \sum_{p=0}^{\infty} \theta_n^p \alpha \xi_{-p}^2 \right) \leq (1-\theta_n) \sum_{p=0}^{\infty} \theta_n^p \mathbb{E} e^{\alpha \xi_{-p}^2} = \mathbb{E} e^{\alpha \xi_0^2}.$$

Summarizing the previous estimates we obtain

$$\mathbb{P} \left( |Z_{0,l,n} - Z_{n-l,l,n}| \geq \frac{rb_n \sqrt{n-l}}{\sqrt{1-\theta_n^2}} \right) \leq 4 \exp \left( -\lambda_n r \frac{b_n \sqrt{n-l}}{2\sqrt{1-\theta_n^2}} \right) \mathbb{E} e^{\alpha \xi_0^2},$$

which yields the desired result by using the assumption

$$b_n \rightarrow \infty, \quad \frac{\sqrt{n}(1-\theta_n)^2}{b_n} \rightarrow \infty.$$

□

**3.2. Some properties of the sequence  $\{U_{k,l,m,n}\}$ .** For all  $n \geq 1$ ,  $0 \leq l \leq M$ ,  $1 \leq k \leq n-l$ ,  $m > 2M$ , set

$$X_{k-1,m,n} = \xi_{k-1} + \theta_n \xi_{k-2} + \cdots + \theta_n^{m-2} \xi_{k-m+1} = \sum_{j=0}^{m-2} \theta_n^j \xi_{k-1-j},$$

and

$$\begin{aligned}
U_{k,l,m,n} &= \theta_n X_{k+l-1,m,n} \xi_k + \xi_{k+l} X_{k-1,m,n} \theta_n + \xi_{k+l} \xi_k - \theta_n^l \mathbb{E} \xi_0^2 \\
&= \sum_{j=1}^{m-1} \theta_n^j \xi_{k+l-j} \xi_k + \sum_{j=1}^{m-1} \xi_{k+l} \xi_{k-j} \theta_n^j + \xi_{k+l} \xi_k - \theta_n^l \mathbb{E} \xi_0^2.
\end{aligned} \tag{3.5}$$

For any  $n, l$ , it is easy to see that  $\{U_{k,l,m,n}\}_{1 \leq k \leq n-l}$  is a strictly stationary sequence with  $m+l$ -dependent structure. Furthermore, the sequence  $\{U_{k,l,m,n}\}_{1 \leq k \leq n-l}$  has the following properties.

**Proposition 3.1.** i) For any  $0 \leq l \leq M$ ,  $1 \leq k \leq n-l$ ,

$$\mathbb{E}(U_{k,l,m,n}) = 0. \quad (3.6)$$

ii) If  $k \neq i$ ,

$$\mathbb{E}(U_{k,0,m,n}U_{i,0,m,n}) = 0. \quad (3.7)$$

iii) If  $l \neq 0$ ,  $k > i$ ,

$$\mathbb{E}(U_{k,l,m,n}U_{i,l,m,n}) = \theta_n^{2l}(\mathbb{E}\xi_0^2)^2 \left( 1_{A_1} + \sum_{q=0}^{m-1-2l} \theta_n^{2q} 1_{A_2} \right) \quad (3.8)$$

where the sets  $A_1, A_2$  are defined by  $A_1 = \{i+l > k\}$ ,  $A_2 = \{i+l = k\}$ .

iv) If  $l \neq 0$ ,

$$\mathbb{E}(U_{k,l,m,n}^2) = \left( \theta_n^{2l} \mathbb{E}\xi_0^4 + \left( 1 - 2\theta_n^{2l} + 2 \sum_{j=1}^{m-1} \theta_n^{2j} \right) (\mathbb{E}\xi_0^2)^2 \right). \quad (3.9)$$

v)

$$\mathbb{E}(U_{k,0,m,n}^2) = \mathbb{E}\xi_0^4 + (\mathbb{E}\xi_0^2)^2 \left[ 4 \sum_{j=1}^{m-1} \theta_n^{2j} - 1 \right]. \quad (3.10)$$

*Proof.* Without loss of generality, we can assume that  $i < k$  and for any  $k$ , let

$$\mathcal{F}_k := \sigma(\xi_i; -\infty < i \leq k).$$

**Proof of i)** The claim (3.6) is easy to be obtained by the properties of conditional expectation.

**Proof of ii)** Since  $\mathbb{E}(U_{k,0,m,n}|\mathcal{F}_{k-1}) = 0$ , and  $U_{i,0,m,n}$  is measurable with respect to  $\mathcal{F}_{k-1}$ , then we have

$$\mathbb{E}(U_{k,0,m,n}U_{i,0,m,n}) = \mathbb{E}[U_{i,0,m,n}\mathbb{E}(U_{k,0,m,n}|\mathcal{F}_{k-1})] = 0.$$

**Proof of iii)** Let

$$\Delta_{1,k,l} := \sum_{j=1}^{m-1} \theta_n^j \xi_{k+l-j} \xi_k, \quad \Delta_{2,k,l} := \sum_{j=1}^{m-1} \xi_{k+l} \xi_{k-j} \theta_n^j.$$

then it is easy to check that  $U_{i,l,m,n}$  is measurable with respect to  $\mathcal{F}_{k+l-1}$  and

$$\mathbb{E}(U_{k,l,m,n}|\mathcal{F}_{k+l-1}) = \Delta_{1,k,l} - \theta_n^l \mathbb{E}\xi_0^2,$$

So we have

$$\mathbb{E}(U_{k,l,m,n}U_{i,l,m,n}) = \mathbb{E}[U_{i,l,m,n}(\Delta_{1,k,l} - \theta_n^l \mathbb{E}\xi_0^2)] = \mathbb{E}(U_{i,l,m,n}\Delta_{1,k,l}).$$

Next we need to calculate the following four terms:

$$(1) \mathbb{E}(\Delta_{1,k,l}\Delta_{1,i,l}), \quad (2) \mathbb{E}(\Delta_{1,k,l}\Delta_{2,i,l}), \quad (3) \mathbb{E}(\Delta_{1,k,l}\xi_{i+l}\xi_i), \quad (4) \mathbb{E}(\Delta_{1,k,l})\theta_n^l \mathbb{E}\xi_0^2.$$

First, we can observe that

- when  $i + l > k$ , then there exists  $1 \leq j, q \leq m - 1$  such that

$$\mathbb{E}(\xi_{k+l-j}\xi_k\xi_{i+l-q}\xi_i) \neq 0;$$

- when  $i + l = k$ , then there exists  $1 \leq j, q \leq m - 1$  such that

$$\mathbb{E}(\xi_{k+l-j}\xi_k\xi_{i+l}\xi_{i-q}) \neq 0;$$

- when  $i + l = k$ , then there exists  $1 \leq j \leq m - 1$  such that

$$\mathbb{E}(\xi_{i+l}\xi_i\xi_{k+l-j}\xi_k) \neq 0.$$

Let  $A_1 = \{i + l > k\}$ ,  $A_2 = \{i + l = k\}$ . Therefore, we have

$$\mathbb{E}(\Delta_{1,k,l}\Delta_{1,i,l}) = \mathbb{E}\left(\sum_{j=1}^{m-1}\theta_n^j\xi_{k+l-j}\xi_k\right)\left(\sum_{q=1}^{m-1}\theta_n^q\xi_{i+l-q}\xi_i\right) = (\theta_n^l\mathbb{E}\xi_0^2)^2(1 + 1_{A_1}),$$

where we take  $j = l = q$  and use the fact that under the case  $i + l > k$ , we may choose  $j = k + l - i$ ,  $q = i + l - k$ . Similarly, we have

$$\mathbb{E}(\Delta_{1,k,l}\Delta_{2,i,l}) = \mathbb{E}\left(\sum_{j=1}^{m-1}\theta_n^j\xi_{k+l-j}\xi_k\right)\left(\sum_{q=1}^{m-1}\xi_{i+l}\xi_{i-q}\theta_n^q\right) = (\mathbb{E}\xi_0^2)^2 1_{A_2} \sum_{q=1}^{m-1-2l}\theta_n^{2q+2l},$$

$$\mathbb{E}(\Delta_{1,k,l}\xi_{i+l}\xi_i) = \mathbb{E}\left(\xi_{i+l}\xi_i\sum_{j=1}^{m-1}\theta_n^j\xi_{k+l-j}\xi_k\right) = (\theta_n^l\mathbb{E}\xi_0^2)^2 1_{A_2}$$

and

$$\mathbb{E}(\Delta_{1,k,l})\theta_n^l\mathbb{E}\xi_0^2 = (\theta_n^l\mathbb{E}\xi_0^2)^2.$$

From the above discussion and the definition of  $U_{i,l,m,n}$ , the proof of iii) is completed.

**Proofs of iv) and v)** Since

$$\begin{aligned} \mathbb{E}U_{k,l,m,n}^2 &= \mathbb{E}\left(\sum_{j=1}^{m-1}\theta_n^j\xi_{k+l-j}\xi_k + \sum_{j=1}^{m-1}\xi_{k+l}\xi_{k-j}\theta_n^j + \xi_{k+l}\xi_k - \theta_n^l\mathbb{E}\xi_0^2\right)^2 \\ &=: \mathbb{E}(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4)^2, \end{aligned}$$



then it is easy to see

$$\begin{aligned}
\mathbb{E}(\Delta_1 \Delta_3) &= \mathbb{E}(\Delta_2 \Delta_3) = \mathbb{E}(\Delta_2 \Delta_4) = 0, \\
\mathbb{E}\Delta_1^2 &= \begin{cases} \theta_n^{2l} \mathbb{E}\xi_0^4 + (\mathbb{E}\xi_0^2)^2 \left( \sum_{j=1}^{m-1} \theta_n^{2j} - \theta_n^{2l} \right), & l \neq 0 \\ (\mathbb{E}\xi_0^2)^2 \sum_{j=1}^{m-1} \theta_n^{2j}, & l = 0, \end{cases} \\
\mathbb{E}(\Delta_1 \Delta_4) &= \begin{cases} -\theta_n^{2l} (\mathbb{E}\xi_0^2)^2, & l \neq 0 \\ 0, & l = 0, \end{cases} \\
\mathbb{E}(\Delta_3^2) &= \begin{cases} (\mathbb{E}\xi_0^2)^2, & l \neq 0 \\ \mathbb{E}\xi_0^4, & l = 0, \end{cases} \\
\mathbb{E}(\Delta_1 \Delta_2) &= \begin{cases} 0, & l \neq 0 \\ (\mathbb{E}\xi_0^2)^2 \sum_{j=1}^{m-1} \theta_n^{2j}, & l = 0, \end{cases} \\
\mathbb{E}(\Delta_3 \Delta_4) &= \begin{cases} 0, & l \neq 0 \\ -(\mathbb{E}\xi_0^2)^2, & l = 0. \end{cases} \\
\mathbb{E}(\Delta_4^2) &= \theta_n^{2l} (\mathbb{E}\xi_0^2)^2, \quad \mathbb{E}(\Delta_2^2) = (\mathbb{E}\xi_0^2)^2 \sum_{j=1}^{m-1} \theta_n^{2j}, \quad \forall 0 \leq l \leq M,
\end{aligned}$$

which yields the desired results.  $\square$

**Proposition 3.2.** *Let  $1 \leq i < k$  and  $0 \leq l, q \leq M$ .*

- (a) *If  $l \neq 0$ , then  $\mathbb{E}(U_{i,0,m,n} U_{k,l,m,n}) = 0$ .*
- (b) *If  $l \neq 0$ , then*

$$\mathbb{E}(U_{k,0,m,n} U_{i,l,m,n}) = 2\theta_n^l (\mathbb{E}\xi_0^2)^2 \left( 1_{A_1} + 1_{A_2} \sum_{j=0}^{m-1-l} \theta_n^{2j} \right)$$

*where the events  $A_1, A_2$  are defined in Proposition 3.1.*

- (c) *If  $0 < l < q$ , then*

$$\mathbb{E}(U_{i,l,m,n} U_{k,q,m,n}) = \theta_n^{l+q} (\mathbb{E}\xi_0^2)^2 \left( 1_{A_1} + 1_{A_2} \sum_{j=0}^{m-1-(l+q)} \theta_n^{2j} \right).$$

- (d) *If  $0 < q < l$ , then*

$$\begin{aligned}
\mathbb{E}(U_{i,l,m,n} U_{k,q,m,n}) &= \theta_n^{l+q} (\mathbb{E}\xi_0^2)^2 \left( 1_{A_1} + 1_{A_2} \sum_{j=0}^{m-1-(l+q)} \theta_n^{2j} \right) \\
&\quad + \theta_n^{l-q} (\mathbb{E}\xi_0^2)^2 \left( 1_{E_1} + 1_{E_2} \sum_{j=0}^{m-1-(l-q)} \theta_n^{2j} \right)
\end{aligned}$$

*where the events  $E_1, E_2$  are defined by*

$$E_1 = \{i + l > k + q\}, \quad E_2 = \{i + l = k + q\}.$$

*Proof.* The proofs of the proposition are similar to the one of Proposition 3.1.

**Proof of (a)** Since  $U_{i,0,m,n}$  is measurable with respect to  $\mathcal{F}_i$ , then we have for  $i < k$ ,

$$\begin{aligned} \mathbb{E}(U_{i,0,m,n}U_{k,l,m,n}) &= \mathbb{E}[U_{i,0,m,n}\mathbb{E}(U_{k,l,m,n}|\mathcal{F}_i)] \\ &= \mathbb{E}\left[U_{i,0,m,n}\left(\sum_{j=1}^{m-1}\theta_n^j\xi_{k+l-j}\xi_k - \theta_n^l\mathbb{E}\xi_0^2\right)\right] = 0. \end{aligned}$$

**Proof of (b)** Since  $i < k$ ,

$$\begin{aligned} \mathbb{E}(U_{k,0,m,n}U_{i,l,m,n}) &= \mathbb{E}\left\{\left(2\sum_{p=1}^{m-1}\theta_n^p\xi_{k-p}\xi_k + \xi_k^2 - \mathbb{E}\xi_0^2\right)\right. \\ &\quad \times \left.\left(\sum_{j=1}^{m-1}\theta_n^j\xi_{i+l-j}\xi_i + \sum_{j=1}^{m-1}\xi_{i+l}\xi_{i-j}\theta_n^j + \xi_{i+l}\xi_i - \theta_n^l\mathbb{E}\xi_0^2\right)\right\} \\ &=: \mathbb{E}(\Delta_1 + \Delta_2 + \Delta_3)(\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4), \end{aligned}$$

then it is easy to check that

$$\mathbb{E}\Delta_1\Gamma_4 = \mathbb{E}\Delta_2\Gamma_2 = \mathbb{E}\Delta_2\Gamma_3 = \mathbb{E}\Delta_3\Gamma_2 = \mathbb{E}\Delta_3\Gamma_3 = 0$$

and

$$\mathbb{E}\Delta_2\Gamma_1 = \mathbb{E}\Delta_3\Gamma_4 = \theta_n^l(\mathbb{E}\xi_0^2)^2, \quad \mathbb{E}\Delta_2\Gamma_4 = \mathbb{E}\Delta_3\Gamma_1 = -\theta_n^l(\mathbb{E}\xi_0^2)^2.$$

Furthermore, we have

- when  $k < i + l$ , then  $\mathbb{E}\Delta_1\Gamma_1 = 2\theta_n^l(\mathbb{E}\xi_0^2)^2$ ;
- when  $k = i + l$ , then  $\mathbb{E}\Delta_1\Gamma_2 = 2\sum_{j=1}^{m-1-l}\theta_n^{l+2j}(\mathbb{E}\xi_0^2)^2$ ;
- when  $k = i + l$ , then  $\mathbb{E}\Delta_1\Gamma_3 = 2\theta_n^l(\mathbb{E}\xi_0^2)^2$ .

So the desired result (b) is obtained.

**Proof of (c)** Since  $i < k$  and  $l < q$ , then  $U_{i,l,m,n}$  is measurable with respect to  $\mathcal{F}_{k+q-1}$ . Hence we have

$$\begin{aligned} \mathbb{E}(U_{i,l,m,n}U_{k,q,m,n}) &= \mathbb{E}[U_{i,l,m,n}\mathbb{E}(U_{k,q,m,n}|\mathcal{F}_{k+q-1})] \\ &= \mathbb{E}\left[U_{i,l,m,n}\left(\sum_{p=1}^{m-1}\theta_n^p\xi_{k+q-p}\xi_k - \theta_n^q\mathbb{E}\xi_0^2\right)\right] \\ &= \mathbb{E}\left[\left(\sum_{j=1}^{m-1}\theta_n^j\xi_{i+l-j}\xi_i + \sum_{j=1}^{m-1}\xi_{i+l}\xi_{i-j}\theta_n^j + \xi_{i+l}\xi_i - \theta_n^l\mathbb{E}\xi_0^2\right)\sum_{p=1}^{m-1}\theta_n^p\xi_{k+q-p}\xi_k\right]. \end{aligned}$$

By the similar discussions as (b), we have

•

$$\mathbb{E}\left(\sum_{j=1}^{m-1}\theta_n^j\xi_{i+l-j}\xi_i\right)\left(\sum_{p=1}^{m-1}\theta_n^p\xi_{k+q-p}\xi_k\right) = \theta_n^{l+q}(\mathbb{E}\xi_0^2)^2(1 + 1_{A_1});$$

- when  $i + l = k$ , we have

$$\mathbb{E} \left( \sum_{j=1}^{m-1} \xi_{i+l} \xi_{i-j} \theta_n^j \right) \left( \sum_{p=1}^{m-1} \theta_n^p \xi_{k+q-p} \xi_k \right) = \sum_{j=1}^{m-1-(l+q)} \theta_n^{l+q+2j} (\mathbb{E} \xi_0^2)^2;$$

- when  $i + l = k$ , we have

$$\mathbb{E} \left( \xi_{i+l} \xi_i \sum_{p=1}^{m-1} \theta_n^p \xi_{k+q-p} \xi_k \right) = \theta_n^{l+q} (\mathbb{E} \xi_0^2)^2;$$

- for the last term,

$$-\theta_n^l \mathbb{E} \xi_0^2 \mathbb{E} \left( \sum_{p=1}^{m-1} \theta_n^p \xi_{k+q-p} \xi_k \right) = -\theta_n^{l+q} (\mathbb{E} \xi_0^2)^2.$$

From the above discussion, the proof of (c) is completed.

**Proof of (d)** Since  $i < k$  and  $q < l$ , and

$$\begin{aligned} \mathbb{E}(U_{i,l,m,n} U_{k,q,m,n}) &= \mathbb{E} \left\{ \left( \sum_{j=1}^{m-1} \theta_n^j \xi_{i+l-j} \xi_i + \sum_{j=1}^{m-1} \xi_{i+l} \xi_{i-j} \theta_n^j + \xi_{i+l} \xi_i - \theta_n^l \mathbb{E} \xi_0^2 \right) \right. \\ &\quad \times \left. \left( \sum_{p=1}^{m-1} \theta_n^p \xi_{k+q-p} \xi_k + \sum_{p=1}^{m-1} \xi_{k+q} \xi_{k-p} \theta_n^p + \xi_{k+q} \xi_k - \theta_n^q \mathbb{E} \xi_0^2 \right) \right\} \\ &=: \mathbb{E}(\hat{\Delta}_1 + \hat{\Delta}_2 + \hat{\Delta}_3 + \hat{\Delta}_4)(\hat{\Gamma}_1 + \hat{\Gamma}_2 + \hat{\Gamma}_3 + \hat{\Gamma}_4), \end{aligned}$$

then it is easy to see that

$$\mathbb{E} \hat{\Delta}_1 \hat{\Gamma}_3 = \mathbb{E} \hat{\Delta}_2 \hat{\Gamma}_3 = \mathbb{E} \hat{\Delta}_3 \hat{\Gamma}_3 = \mathbb{E} \hat{\Delta}_4 \hat{\Gamma}_3 = \mathbb{E} \hat{\Delta}_2 \hat{\Gamma}_4 = \mathbb{E} \hat{\Delta}_3 \hat{\Gamma}_4 = \mathbb{E} \hat{\Delta}_4 \hat{\Gamma}_2 = 0$$

and

$$\mathbb{E} \hat{\Delta}_1 \hat{\Gamma}_4 = \mathbb{E} \hat{\Delta}_4 \hat{\Gamma}_1 = -\mathbb{E} \hat{\Delta}_4 \hat{\Gamma}_4 = -\theta_n^{l+q} (\mathbb{E} \xi_0^2)^2.$$

In addition, we have

$$\mathbb{E} \hat{\Delta}_1 \hat{\Gamma}_1 = \theta_n^{l+q} (\mathbb{E} \xi_0^2)^2 (1 + 1_{A_1}),$$

$$\mathbb{E} \hat{\Delta}_2 \hat{\Gamma}_1 = \theta_n^{l+q} (\mathbb{E} \xi_0^2)^2 \sum_{j=1}^{m-1-(l+q)} \theta_n^{2j} 1_{A_2}$$

and

$$\mathbb{E} \hat{\Delta}_3 \hat{\Gamma}_1 = \theta_n^{l+q} (\mathbb{E} \xi_0^2)^2 1_{A_2}.$$

Similarly, we can observe that

- when  $i + l > k + q$ , then there exists  $1 \leq j, q \leq m - 1$  such that

$$\mathbb{E}(\xi_{k+q} \xi_{k-p} \xi_{i+l-j} \xi_i) \neq 0;$$

- when  $i + l = k + q$ , then there exists  $1 \leq j, q \leq m - 1$  such that

$$\mathbb{E}(\xi_{k+q} \xi_{k-p} \xi_{i+l} \xi_{i-j}) \neq 0;$$

- when  $i + l = k + q$ , then there exists  $1 \leq p \leq m - 1$  such that

$$\mathbb{E}(\xi_{i+l} \xi_i \xi_{k+q} \xi_{k-p}) \neq 0.$$

Hence we have

$$\begin{aligned}\mathbb{E}\hat{\Delta}_1\hat{\Gamma}_2 &= \theta_n^{l-q}(\mathbb{E}\xi_0^2)^2 1_{E_1}, \\ \mathbb{E}\hat{\Delta}_2\hat{\Gamma}_2 &= \theta_n^{l-q}(\mathbb{E}\xi_0^2)^2 \sum_{j=1}^{m-1-(l-q)} \theta_n^{2j} 1_{E_2}\end{aligned}$$

and

$$\mathbb{E}\hat{\Delta}_3\hat{\Gamma}_2 = \theta_n^{l-q}(\mathbb{E}\xi_0^2)^2 1_{E_2}.$$

Combining the above results, we complete the proof of (d).  $\square$

**Proposition 3.3.** *When  $i = k$ , we have*

(1) *If  $0 < l < q$ , we have*

$$\mathbb{E}(U_{i,l,m,n}U_{i,q,m,n}) = \theta_n^{l+q}(\mathbb{E}\xi_0^4) - 2\theta_n^{l+q}(\mathbb{E}\xi_0^2)^2 + \theta_n^{q-l}(\mathbb{E}\xi_0^2)^2 \sum_{j=0}^{m-1-(q-l)} \theta_n^{2j}.$$

(2) *If  $0 < q < l$ , we have*

$$\mathbb{E}(U_{i,l,m,n}U_{i,q,m,n}) = \theta_n^{l+q}(\mathbb{E}\xi_0^4) - 2\theta_n^{l+q}(\mathbb{E}\xi_0^2)^2 + \theta_n^{l-q}(\mathbb{E}\xi_0^2)^2 \sum_{p=0}^{m-1-(l-q)} \theta_n^{2p}.$$

(3) *If  $l = q \neq 0$ , we have*

$$\mathbb{E}(U_{i,l,m,n}^2) = \theta_n^{2l}(\mathbb{E}\xi_0^4) - 2\theta_n^{2l}(\mathbb{E}\xi_0^2)^2 + (\mathbb{E}\xi_0^2)^2 \left( 2 \sum_{j=1}^{m-1} \theta_n^{2j} + 1 \right).$$

(4) *If  $q > 0$ , we have*

$$\mathbb{E}(U_{i,0,m,n}U_{i,q,m,n}) = \theta_n^q(\mathbb{E}\xi_0^4) - \theta_n^q(\mathbb{E}\xi_0^2)^2 + 2(\mathbb{E}\xi_0^2)^2 \theta_n^q \sum_{j=1}^{m-1-q} \theta_n^{2j}.$$

*Proof.* From

$$\begin{aligned}\mathbb{E}(U_{i,l,m,n}U_{i,q,m,n}) &= \mathbb{E} \left\{ \left( \sum_{j=1}^{m-1} \theta_n^j \xi_{i+l-j} \xi_i + \sum_{j=1}^{m-1} \xi_{i+l} \xi_{i-j} \theta_n^j + \xi_{i+l} \xi_i - \theta_n^l \mathbb{E}\xi_0^2 \right) \right. \\ &\quad \times \left. \left( \sum_{p=1}^{m-1} \theta_n^p \xi_{i+q-p} \xi_i + \sum_{p=1}^{m-1} \xi_{i+q} \xi_{i-p} \theta_n^p + \xi_{i+q} \xi_i - \theta_n^q \mathbb{E}\xi_0^2 \right) \right\} \\ &=: \mathbb{E}(\tilde{\Delta}_1 + \tilde{\Delta}_2 + \tilde{\Delta}_3 + \tilde{\Delta}_4)(\tilde{\Gamma}_1 + \tilde{\Gamma}_2 + \tilde{\Gamma}_3 + \tilde{\Gamma}_4),\end{aligned}$$

we know that for any  $0 < l, q \leq M$ ,

$$\mathbb{E}\tilde{\Delta}_1\tilde{\Gamma}_2 = \mathbb{E}\tilde{\Delta}_2\tilde{\Gamma}_1 = \mathbb{E}\tilde{\Delta}_2\tilde{\Gamma}_3 = \mathbb{E}\tilde{\Delta}_2\tilde{\Gamma}_4 = \mathbb{E}\tilde{\Delta}_3\tilde{\Gamma}_2 = \mathbb{E}\tilde{\Delta}_3\tilde{\Gamma}_4 = \mathbb{E}\tilde{\Delta}_4\tilde{\Gamma}_2 = \mathbb{E}\tilde{\Delta}_4\tilde{\Gamma}_3 = 0,$$

$$\mathbb{E}\tilde{\Delta}_1\tilde{\Gamma}_4 = \mathbb{E}\tilde{\Delta}_4\tilde{\Gamma}_1 = -\mathbb{E}\tilde{\Delta}_4\tilde{\Gamma}_4 = -\theta_n^{l+q}(\mathbb{E}\xi_0^2)^2$$

and

$$\begin{aligned}
\mathbb{E}\tilde{\Delta}_1\tilde{\Gamma}_1 &= \begin{cases} \theta_n^{2l}\mathbb{E}\xi_0^4 + (\mathbb{E}\xi_0^2)^2 \left( \sum_{j=1}^{m-1} \theta_n^{2j} - \theta_n^{2l} \right), & l = q \\ \theta_n^{l+q}\mathbb{E}\xi_0^4 + (\mathbb{E}\xi_0^2)^2 \left( \sum_{p=1}^{m-1-(l-q)} \theta_n^{l-q+2p} - \theta_n^{l+q} \right), & l > q, \\ \theta_n^{l+q}\mathbb{E}\xi_0^4 + (\mathbb{E}\xi_0^2)^2 \left( \sum_{p=1}^{m-1-(q-l)} \theta_n^{q-l+2p} - \theta_n^{l+q} \right), & l < q \end{cases} \\
\mathbb{E}\tilde{\Delta}_1\tilde{\Gamma}_3 &= \begin{cases} \theta_n^{l-q}(\mathbb{E}\xi_0^2)^2, & l > q \\ 0, & l \leq q \end{cases}, \\
\mathbb{E}\tilde{\Delta}_3\tilde{\Gamma}_1 &= \begin{cases} \theta_n^{q-l}(\mathbb{E}\xi_0^2)^2, & q > l \\ 0, & q \leq l \end{cases}, \\
\mathbb{E}\tilde{\Delta}_2\tilde{\Gamma}_2 &= \begin{cases} (\mathbb{E}\xi_0^2)^2 \sum_{j=1}^{m-1} \theta_n^{2j}, & l = q \\ 0, & l \neq q \end{cases}, \\
\mathbb{E}\tilde{\Delta}_3\tilde{\Gamma}_3 &= \begin{cases} (\mathbb{E}\xi_0^2)^2, & q = l \\ 0, & q \neq l \end{cases}.
\end{aligned}$$

So the results (1)-(3) hold. Furthermore, (4) can be obtained by the following observation

$$\mathbb{E}(U_{i,0,m,n}U_{i,q,m,n}) = \mathbb{E}[U_{i,0,m,n}\mathbb{E}(U_{i,q,m,n}|\mathcal{F}_{i+q-1})] = \mathbb{E}\left(U_{i,0,m,n} \sum_{p=1}^{m-1} \theta_n^p \xi_{i+q-p} \xi_i\right).$$

□

### 3.3. Moderate deviation for $m$ -dependent sequence with unbounded $m$ .

Before giving our proofs of the main results, it is necessary to give the following moderate deviation principle for  $m$ -dependent random variables with unbounded  $m$ . For the readability of the paper, we postpone its proof to Appendix.

**Lemma 3.3.** *For each  $n = 1, 2, \dots$ , let  $m = m(n)$  be specified and suppose that  $\{X_{1,n}, \dots, X_{n,n}\}$  be a sequence of strict stationary  $m$ -dependent random variables with zero means. Moreover, we assume the following conditions hold:*

- (A) *there exists a positive  $0 < \gamma < 1$  such that the moderate deviation scale  $(b_n)$  satisfies*

$$b_n \rightarrow \infty, \quad \frac{b_n m^{1+\gamma}}{\sqrt{n}} \rightarrow 0;$$

- (B) *for some  $M > 0$ ,*

$$\frac{n}{b_n^2 m} \int_M^\infty e^x \mathbb{P}\left(|X_{1,n}| \geq \frac{\sqrt{n}x}{b_n m}\right) dx \rightarrow 0;$$

- (C) *for any  $\varepsilon > 0$ ,*

$$\left(\frac{\sqrt{n}}{b_n}\right)^{2+\frac{2}{1+\gamma}} \mathbb{P}\left(|X_{1,n}| > \varepsilon \left(\frac{\sqrt{n}}{b_n}\right)^{1-\frac{1}{1+\gamma}}\right) \rightarrow 0;$$

- (D) *there exists a constant  $0 < \sigma^2 < \infty$ , such that*

$$\lim_{n \rightarrow \infty} m^{-1} \text{Var}(X_{1,n} + \dots + X_{m,n}) = \sigma^2$$

and

$$\lim_{n \rightarrow \infty} m^{-1} \sum_{i=1}^m i \mathbb{E}(X_{1,n} X_{i+1,n}) = 0.$$

Then for any  $\lambda \in \mathbb{R}$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{E} \exp \left( \lambda \frac{b_n}{\sqrt{n}} \sum_{i=1}^n X_{i,n} \right) = \frac{\lambda^2 \sigma^2}{2}.$$

Furthermore, by the Gärtner-Ellis theorem (see [4]), for any  $r > 0$ , we get

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{1}{b_n \sqrt{n}} \left| \sum_{i=1}^n X_{i,n} \right| \geq r \right) = -\frac{r^2}{2\sigma^2}.$$

**Remark 3.1.** In [3], Chen established the moderate deviation for  $m$ -dependent random vectors with fixed parameter  $m$ . Recently, Miao and Yang [13] proved the following moderate deviation, which extended Chen's result from fixed  $m$  to unbounded  $m$  for  $\mathbb{R}$ -valued  $m$ -dependent sequence:

Assume that

$$\sup_n \mathbb{E} \exp\{\alpha |X_{1,n}|\} < \infty, \quad \text{for some } \alpha > 0 \quad (3.11)$$

and

$$b_n \rightarrow \infty, \quad \frac{b_n m^2}{\sqrt{n}} \rightarrow 0. \quad (3.12)$$

In addition, if the condition (D) hold, then for any  $\lambda \in \mathbb{R}$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{E} \exp \left( \lambda \frac{b_n}{\sqrt{n}} \sum_{i=1}^n X_{i,n} \right) = \frac{\lambda^2 \sigma^2}{2}.$$

It is easy to see that the condition (3.11) and (3.12) imply the conditions (A), (B) and (C). But, the condition (3.11) is not easy to check in the process of proving our main results, so we need develop a new moderate deviation for  $m$ -dependent sequence with unbounded  $m$ , that is, Lemma 3.3.

#### 4. PROOF OF THEOREM 2.1

**4.1. Asymptotic term and moderate deviations.** We have the following useful results, based on the properties of the sequence  $\{U_{k,l,m,n}\}_{1 \leq k \leq n-l}$ .

**Corollary 4.1.** Let  $m := m(n)$  denote the subsequence of  $n$  such that  $m(1 - \theta_n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Then we have

$$\lim_{n \rightarrow \infty} \frac{1 - \theta_n^2}{m} \sum_{k=1}^m k \mathbb{E}(U_{1,l,m,n} U_{k+1,l,m,n}) = 0 \quad (4.1)$$

and

$$\lim_{n \rightarrow \infty} \frac{1 - \theta_n^2}{m} \text{Var}(U_{1,l,m,n} + \cdots + U_{m,l,m,n}) = 4(\mathbb{E}\xi_0^2)^2. \quad (4.2)$$

*Proof.* **Case  $l \neq 0$ .** From iii) in Proposition 3.1, it is easy to see

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1 - \theta_n^2}{m} \sum_{k=1}^m k \mathbb{E}(U_{1,l,m,n} U_{k+1,l,m,n}) \\
&= \lim_{n \rightarrow \infty} \frac{1 - \theta_n^2}{m} \sum_{k=1}^l k \mathbb{E}(U_{1,l,m,n} U_{k+1,l,m,n}) \\
&= \lim_{n \rightarrow \infty} \frac{1 - \theta_n^2}{m} \left[ \sum_{k=1}^{l-1} k \mathbb{E}(U_{1,l,m,n} U_{k+1,l,m,n}) + l \mathbb{E}(U_{1,l,m,n} U_{l+1,l,m,n}) \right] \\
&= \lim_{n \rightarrow \infty} \frac{1 - \theta_n^2}{m} \left[ \sum_{k=1}^l k (\theta_n^l \mathbb{E} \xi_0^2)^2 + l (\mathbb{E} \xi_0^2)^2 \sum_{q=1}^{m-1-2l} \theta_n^{2q+2l} \right] = 0.
\end{aligned}$$

Furthermore, since

$$\begin{aligned}
& \text{Var}(U_{1,l,m,n} + \cdots + U_{m,l,m,n}) \\
&= \sum_{k=1}^m \mathbb{E} U_{k,l,m,n}^2 + 2 \sum_{k=1}^{m-1} \sum_{q=k+1}^m \mathbb{E}(U_{k,l,m,n} U_{q,l,m,n}) \\
&= \sum_{k=1}^m \mathbb{E} U_{k,l,m,n}^2 + 2 \sum_{k=1}^{m-l} \sum_{q=k+1}^{k+l} \mathbb{E}(U_{k,l,m,n} U_{q,l,m,n}),
\end{aligned}$$

then, by iii) and iv) in Proposition 3.1, we have

$$\sum_{k=1}^{m-l} \sum_{q=k+1}^{k+l} \mathbb{E}(U_{k,l,m,n} U_{q,l,m,n}) = (m-l) \left( l (\theta_n^l \mathbb{E} \xi_0^2)^2 + (\mathbb{E} \xi_0^2)^2 \sum_{q=1}^{m-1-2l} \theta_n^{2q+2l} \right)$$

and

$$\sum_{k=1}^m \mathbb{E} U_{k,l,m,n}^2 = m \left( \theta_n^{2l} \mathbb{E} \xi_0^4 + \left( 1 - 2\theta_n^{2l} + 2 \sum_{j=1}^{m-1} \theta_n^{2j} \right) (\mathbb{E} \xi_0^2)^2 \right).$$

Hence it follows that

$$\begin{aligned}
& \text{Var}(U_{1,l,m,n} + \cdots + U_{m,l,m,n}) \\
&= m \theta_n^{2l} \mathbb{E} \xi_0^4 + (m + [2(m-l)l - 2m] \theta_n^{2l}) (\mathbb{E} \xi_0^2)^2 \\
&\quad + \left( 2m \sum_{j=1}^{m-1} \theta_n^{2j} + 2(m-l) \theta_n^{2l} \sum_{j=1}^{m-1-2l} \theta_n^{2j} \right) (\mathbb{E} \xi_0^2)^2.
\end{aligned}$$

By the assumption  $(1 - \theta_n)m \rightarrow \infty$  (which implies  $\theta_n^m \rightarrow 0$ ), we have

$$\lim_{n \rightarrow \infty} \frac{1 - \theta_n^2}{m} \text{Var}(U_{1,l,m,n} + \cdots + U_{m,l,m,n}) = 4(\mathbb{E} \xi_0^2)^2.$$

**Case  $l = 0$ .** From ii) in Proposition 3.1, we have

$$\lim_{n \rightarrow \infty} \frac{1 - \theta_n^2}{m} \sum_{k=1}^m k \mathbb{E}(U_{1,0,m,n} U_{k+1,0,m,n}) = 0$$

and

$$\text{Var}(U_{1,0,m,n} + \cdots + U_{m,0,m,n}) = \sum_{k=1}^m \mathbb{E} U_{k,0,m,n}^2.$$

By v) in Proposition 3.1, it follows that

$$\mathbb{E}U_{k,0,m,n}^2 = \mathbb{E}\xi_0^4 + (\mathbb{E}\xi_0^2)^2 \left[ 4 \sum_{j=1}^{m-1} \theta_n^{2j} - 1 \right] \quad (4.3)$$

then we have

$$\lim_{n \rightarrow \infty} \frac{1 - \theta_n^2}{m} \text{Var}(U_{1,l,m,n} + \cdots + U_{m,l,m,n}) = 4(\mathbb{E}\xi_0^2)^2.$$

□

Before giving the following proposition, we need to mention the **claim**: owing to the conditions

$$n(1 - \theta_n) \rightarrow \infty \quad \text{and} \quad \frac{b_n}{(1 - \theta_n)^2 \sqrt{n}} \rightarrow 0,$$

there must exist a subsequence  $m = m(n)$  such that

$$m(1 - \theta_n) \rightarrow \infty, \quad \frac{m(1 - \theta_n)}{|\log(1 - \theta_n)|} \rightarrow \infty \quad \text{and} \quad \frac{b_n m^{5/3}}{\sqrt{n}} \rightarrow 0. \quad (4.4)$$

For instance, we can take

$$m = (1 - \theta_n)^{-6/5}.$$

Now, based on the above preparations, we have the following result.

**Proposition 4.1.** *Under the assumptions of Theorem 2.1, for any  $\lambda \in \mathbb{R}$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{E} \exp \left\{ \lambda \frac{b_n \sqrt{1 - \theta_n^2}}{\sqrt{n - l}} \sum_{k=1}^{n-l} U_{k,l,m,n} \right\} = 2\lambda^2 (\mathbb{E}\xi_0^2)^2, \quad 0 \leq l \leq M,$$

where the sequence  $\{m\}$  satisfies the properties in (4.4). Furthermore, for any  $r > 0$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{\sqrt{1 - \theta_n^2}}{b_n \sqrt{n - l}} \left| \sum_{k=1}^{n-l} U_{k,l,m,n} \right| \geq r \right) = -\frac{r^2}{8(\mathbb{E}\xi_0^2)^2}, \quad 0 \leq l \leq M.$$

*Proof.* Set

$$K_m(\theta_n) = \sum_{j=1}^{m-1} \theta_n^j$$

then it is easy to see that

$$\lim_{n \rightarrow \infty} \theta_n^{2m} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} (1 - \theta_n^2) K_m(\theta_n) = 2. \quad (4.5)$$

For every  $0 \leq l \leq M$ , let

$$X_{1,l,n} := \sqrt{1 - \theta_n^2} U_{1,l,m,n}$$

then by the properties of  $U_{1,l,m,n}$ , we have

$$\lim_{n \rightarrow \infty} \mathbb{E}X_{1,0,n}^2 = 4(\mathbb{E}\xi_0^2)^2, \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbb{E}X_{1,l,n}^2 = 2(\mathbb{E}\xi_0^2)^2, \quad l \neq 0.$$

Furthermore, from Corollary 4.1, for any  $0 \leq l \leq M$ , it follows that

$$\lim_{n \rightarrow \infty} \frac{1}{m} \text{Var}(X_{1,l,n} + \cdots + X_{m,l,n}) = 4(\mathbb{E}\xi_0^2)^2 \quad (4.6)$$



and

$$\lim_{n \rightarrow \infty} \frac{1}{m} \sum_{k=1}^m k \mathbb{E}(X_{1,l,n} X_{k+1,l,n}) = 0. \quad (4.7)$$

Next, we need to check the conditions (B) and (C) of Lemma 3.3 for the random variable  $X_{1,l,n}$ , namely, for any  $M > 0$ ,

$$\frac{n}{b_n^2 m} \int_M^\infty e^x \mathbb{P} \left( |X_{1,l,n}| \geq \frac{\sqrt{n}x}{b_n m} \right) dx \rightarrow 0 \quad (4.8)$$

and for any  $\varepsilon > 0$ ,

$$\left( \frac{\sqrt{n}}{b_n} \right)^{16/5} \mathbb{P} \left( |X_{1,l,n}| > \varepsilon \left( \frac{\sqrt{n}}{b_n} \right)^{2/5} \right) \rightarrow 0, \quad (4.9)$$

where we take  $\gamma = 2/3$ . However, from the definition of  $X_{1,l,n}$ ,

$$X_{1,l,n} = \sqrt{1 - \theta_n^2} \left( \sum_{j=1}^{m-1} \theta_n^j \xi_{1+l-j} \xi_1 + \sum_{j=1}^{m-1} \xi_{1+l} \xi_{1-j} \theta_n^j + \xi_{1+l} \xi_1 - \theta_n^l \mathbb{E} \xi_0^2 \right)$$

it is enough to show that (4.8) and (4.9) hold for the term  $\sqrt{1 - \theta_n^2} \sum_{j=1}^{m-1} \theta_n^j \xi_{1+l-j} \xi_1$ , and the proofs of the others are similar. By the conditions

$$m(1 - \theta_n^2) \rightarrow \infty \quad \text{and} \quad \frac{b_n m^{5/3}}{\sqrt{n}} \rightarrow 0,$$

we know that for all  $n$  sufficient large

$$\frac{\sqrt{n}}{b_n m} \sqrt{1 - \theta_n^2} \geq \frac{\sqrt{n}}{b_n m^{3/2}} \geq \left( \frac{\sqrt{n}}{b_n} \right)^{1/10}.$$

From Cauchy-Schwarz's inequality, we have

$$\begin{aligned} & \mathbb{E} \exp \left\{ \frac{\alpha(1 - \theta_n^2)}{3} \sum_{j=1}^{m-1} \theta_n^j |\xi_{1+l-j} \xi_1| \right\} \\ & \leq \mathbb{E} \exp \left\{ \frac{\alpha(1 - \theta_n^2)}{6} \sum_{j=1}^{m-1} \theta_n^j (\xi_{1+l-j}^2 + \xi_1^2) \right\} \\ & \leq \left( \mathbb{E} \exp \left\{ \frac{\alpha(1 - \theta_n^2)}{3} \sum_{j=1}^{m-1} \theta_n^j \xi_{1+l-j}^2 \right\} \right)^{1/2} \left( \mathbb{E} \exp \left\{ \frac{\alpha(1 - \theta_n^2)}{3} \sum_{j=1}^{m-1} \theta_n^j \xi_1^2 \right\} \right)^{1/2} \\ & \leq \mathbb{E} \exp \left\{ \frac{\alpha(1 - \theta_n^2)}{3} K_m(\theta_n) \xi_1^2 \right\} \leq \mathbb{E} e^{\alpha \xi_1^2}, \end{aligned}$$

so, for any  $M > 0$ ,

$$\begin{aligned}
& \frac{n}{b_n^2 m} \int_M^\infty e^x \mathbb{P} \left( \left| (1 - \theta_n^2) \sum_{j=1}^{m-1} \theta_n^j \xi_{1+l-j} \xi_1 \right| \geq \frac{\sqrt{n}x}{b_n m} \sqrt{(1 - \theta_n^2)} \right) dx \\
& \leq \frac{n}{b_n^2} \int_M^\infty e^x \mathbb{P} \left( \frac{\alpha(1 - \theta_n^2)}{3} \sum_{j=1}^{m-1} \theta_n^j |\xi_{1+l-j} \xi_1| \geq \left( \frac{\sqrt{n}}{b_n} \right)^{1/10} \frac{\alpha x}{3} \right) dx \\
& \leq \mathbb{E} e^{\alpha \xi_1^2} \frac{n}{b_n^2} \int_M^\infty \exp \left\{ - \left[ \left( \frac{\sqrt{n}}{b_n} \right)^{1/10} \frac{\alpha}{3} - 1 \right] x \right\} dx \rightarrow 0.
\end{aligned} \tag{4.10}$$

In addition, from the fact that

$$\frac{(1 - \theta_n^2)^{1/2}}{(\sqrt{n}/b_n)^{-3/10}} \rightarrow \infty,$$

we have, for any  $\varepsilon > 0$ ,

$$\begin{aligned}
& \left( \frac{\sqrt{n}}{b_n} \right)^{16/5} \mathbb{P} \left( \left| (1 - \theta_n^2) \sum_{j=1}^{m-1} \theta_n^j \xi_{1+l-j} \xi_1 \right| > \varepsilon \left( \frac{\sqrt{n}}{b_n} \right)^{2/5} \sqrt{(1 - \theta_n^2)} \right) \\
& \leq \left( \frac{\sqrt{n}}{b_n} \right)^{16/5} \mathbb{P} \left( \left| (1 - \theta_n^2) \sum_{j=1}^{m-1} \theta_n^j \xi_{1+l-j} \xi_1 \right| > \varepsilon \left( \frac{\sqrt{n}}{b_n} \right)^{1/10} \right) \\
& \leq \mathbb{E} e^{\alpha \xi_1^2} \left( \frac{\sqrt{n}}{b_n} \right)^{16/5} \exp \left\{ -\varepsilon \left( \frac{\sqrt{n}}{b_n} \right)^{1/10} \frac{\alpha}{3} \right\} \rightarrow 0.
\end{aligned} \tag{4.11}$$

Therefore, the conditions in Lemma 3.3 are satisfied and the desired results of the proposition can be obtained.  $\square$

**4.2. Exponential approximation.** In this subsection, we shall establish the asymptotic negligibility of the term  $\frac{\sqrt{1-\theta_n^2}}{b_n \sqrt{n-l}} \sum_{k=1}^{n-l} (U_{k,l,m,n} - U_{k,l,n})$  as  $n \rightarrow \infty$ . For all  $p \geq 0$  and  $k \geq 1$ , set

$$W_{k,p} = \xi_k \xi_{k-p}. \tag{4.12}$$

**Lemma 4.1.** *Let the assumptions of Proposition 4.1 hold.*

(1) *There exist  $\alpha_0$  and  $\beta_0$  such that for all  $p \geq 1$ ,  $n \geq 1$  and  $t \geq 0$*

$$\mathbb{P} \left( \max_{j \leq n} \left| \sum_{k=1}^j W_{k,p} \right| \geq t \right) \leq 36 \exp \left( -\frac{t^2}{\alpha_0 n + \beta_0 t} \right). \tag{4.13}$$

(2) *For all  $t > 0$ , there exist  $N \geq 1$ ,  $A, B > 0$  such that, for all  $n \geq N$  and  $0 \leq l \leq M$ ,*

$$\begin{aligned}
& \mathbb{P} \left( \max_{j \leq n-l} \left| \sum_{k=1}^j (U_{k,l,m,n} - U_{k,l,n}) \right| \geq t b_n \frac{\sqrt{n-l}}{\sqrt{1-\theta_n^2}} \right) \\
& \leq 72 \left( 1 - \exp \left( -\frac{b_n^2 t^2}{(At + B) K_n \theta_n^m \sqrt{1-\theta_n^2}} \right) \right)^{-1} \exp \left( -\frac{b_n^2 t^2}{(At + B) K_n \theta_n^m \sqrt{1-\theta_n^2}} \right),
\end{aligned} \tag{4.14}$$

where  $K_n = (1 - \theta_n)^{-2}$  and  $K_n \theta_n^m \sqrt{1 - \theta_n^2} \rightarrow 0$  as  $n \rightarrow \infty$ .

(3) For all  $t > 0$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{\sqrt{1 - \theta_n^2}}{b_n \sqrt{n - l}} \left| \sum_{k=1}^{n-l} (U_{k,l,m,n} - U_{k,l,n}) \right| \geq t \right) = -\infty.$$

*Proof.* (1) This is Lemma 17 in [11].

(2) Firstly, we have

$$\begin{aligned} \left| \sum_{k=1}^j (U_{k,l,m,n} - U_{k,l,n}) \right| &\leq \left| \sum_{k=1}^j \theta_n \xi_k (X_{k+l-1,n} - X_{k+l-1,m,n}) \right| \\ &\quad + \left| \sum_{k=1}^j \theta_n \xi_{k+l} (X_{k-1,n} - X_{k-1,m,n}) \right|. \end{aligned} \quad (4.15)$$

Since

$$X_{k+l-1,n} - X_{k+l-1,m,n} = \theta_n^{m-1} \sum_{p=0}^{\infty} \theta_n^p \xi_{k+l-m-p},$$

we can get

$$\left| \sum_{k=1}^j \theta_n \xi_k (X_{k+l-1,n} - X_{k+l-1,m,n}) \right| \leq \theta_n^m \sum_{p=0}^{\infty} \theta_n^p \left| \sum_{k=1}^j W_{k,m+p-l} \right|.$$

Now it is not difficult to show the fact: for any  $n \geq 1$ ,

$$K_n = \sum_{p=0}^{\infty} (p+1) \theta_n^p = (1 - \theta_n)^{-2}.$$

Hence, by (4.13), we have

$$\begin{aligned} &\mathbb{P} \left( \max_{1 \leq j \leq n-l} \left| \sum_{k=1}^j \theta_n \xi_k (X_{k+l-1,n} - X_{k+l-1,m,n}) \right| > t b_n \frac{\sqrt{n-l}}{2\sqrt{1-\theta_n^2}} \right) \\ &\leq \mathbb{P} \left( \sum_{p=0}^{\infty} (p+1) \frac{\theta_n^p}{p+1} \max_{1 \leq j \leq n-l} \left| \sum_{k=1}^j W_{k,m+p-l} \right| > \sum_{p=0}^{\infty} (p+1) \theta_n^p \frac{t b_n \sqrt{n-l}}{2 K_n \sqrt{1-\theta_n^2} \theta_n^m} \right) \\ &\leq \sum_{p=0}^{\infty} \mathbb{P} \left( \max_{1 \leq j \leq n-l} \left| \sum_{k=1}^j W_{k,m+p-l} \right| > \frac{t b_n (p+1) \sqrt{n-l}}{2 K_n \sqrt{1-\theta_n^2} \theta_n^m} \right) \\ &\leq 36 \sum_{p=0}^{\infty} \exp \left( - \frac{b_n^2 t_{m,p}^2(t)}{\alpha_0 + \beta_0 t_{m,p}(t) b_n / \sqrt{n-l}} \right) \end{aligned} \quad (4.16)$$

where

$$t_{m,p}(t) = \frac{t(p+1)}{2 K_n \theta_n^m \sqrt{1-\theta_n^2}}.$$

By noting that

$$\frac{m(1-\theta_n)}{|\log(1-\theta_n)|} \rightarrow \infty \implies \lim_{n \rightarrow \infty} K_n \theta_n^m \sqrt{1-\theta_n^2} = 0$$

and from the assumption of  $b_n$ , there exist constants  $N \in \mathbb{N}$ ,  $A, B > 0$ , such that for all  $n \geq N$ ,  $l \geq 0$ , and we obtain

$$\frac{t_{m,p}^2(t)}{\alpha_0 + \beta_0 t_{m,p}(t) b_n / \sqrt{n-l}} \geq c(t) \frac{p+1}{K_n \theta_n^m \sqrt{1-\theta_n^2}}, \quad c(t) := \frac{t^2}{At+B}. \quad (4.17)$$

Hence, by (4.16) and (4.17), we get

$$\begin{aligned} & \mathbb{P} \left( \max_{1 \leq j \leq n-l} \left| \sum_{k=1}^j \theta_n \xi_k (X_{k+l-1,n} - X_{k+l-1,m,n}) \right| > t b_n \frac{\sqrt{n-l}}{2\sqrt{1-\theta_n^2}} \right) \\ & \leq 36 \sum_{p=0}^{\infty} \exp \left( -b_n^2 c(t) \frac{p+1}{K_n \theta_n^m \sqrt{1-\theta_n^2}} \right) \\ & = 36 \left( 1 - \exp \left( -\frac{b_n^2 c(t)}{K_n \theta_n^m \sqrt{1-\theta_n^2}} \right) \right)^{-1} \exp \left( -\frac{b_n^2 c(t)}{K_n \theta_n^m \sqrt{1-\theta_n^2}} \right). \end{aligned} \quad (4.18)$$

For the same reason, we can give the estimate of the second term in (4.15), so the proof of (4.14) can be completed..

(3) It follows obviously by (4.14).  $\square$

At last, the proof of Theorem 2.1 can be completed by Lemma 3.1, Proposition 4.1 and (3) in Lemma 4.1.

## 5. PROOF OF THEOREM 2.2

Let  $Y_{k,l,n} = X_{k+l,n} X_{k,n} - \mathbb{E} X_{k+l,n} X_{k,n}$  for any  $1 \leq k \leq n$  and  $0 \leq l \leq M$ . Since

$$\sum_{l=0}^M a_{l,n} \sum_{k=1}^{n-l} Y_{k,l,n} = \sum_{k=1}^{n-M} \sum_{l=0}^M a_{l,n} Y_{k,l,n} + \sum_{l=0}^{M-1} \sum_{k=n-M+1}^{n-l} a_{l,n} Y_{k,l,n}, \quad (5.1)$$

then the desired result is equivalent to showing

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{(1-\theta_n^2)^{3/2}}{b_n \sqrt{n}} \left| \sum_{k=1}^{n-M} \sum_{l=0}^M a_{l,n} Y_{k,l,n} \right| \geq r \right) = -\frac{r^2}{2\Sigma^2} \quad (5.2)$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{(1-\theta_n^2)^{3/2}}{b_n \sqrt{n}} \left| \sum_{l=0}^{M-1} \sum_{k=n-M+1}^{n-l} a_{l,n} Y_{k,l,n} \right| \geq r \right) = -\infty. \quad (5.3)$$

As the similar proof of Theorem 2.1, in order to obtain (5.2), it is enough to show that for any  $\lambda \in \mathbb{R}$ , it follows

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{E} \exp \left\{ \lambda \frac{b_n \sqrt{1-\theta_n^2}}{\sqrt{n}} \sum_{k=1}^{n-M} \sum_{l=0}^M a_{l,n} U_{k,l,m,n} \right\} = \frac{\lambda^2 \Sigma^2}{2}, \quad (5.4)$$

where the sequence  $\{m\}$  satisfies the properties in (4.4). Let

$$\hat{Y}_{k,m,n} := \sum_{l=0}^M a_{l,n} U_{k,l,m,n}, \quad (5.5)$$

then it is easy to see that  $\{\hat{Y}_{k,m,n}\}_{1 \leq k \leq n-M}$  is a strictly stationary sequence with  $m+M$ -dependent structure. Hence from Lemma 3.3, (5.4) is equivalent to proving

$$\lim_{n \rightarrow \infty} \frac{1 - \theta_n^2}{m} \sum_{k=1}^m k \mathbb{E}(\hat{Y}_{1,m,n} \hat{Y}_{k+1,m,n}) = 0 \quad (5.6)$$

and

$$\lim_{n \rightarrow \infty} \frac{1 - \theta_n^2}{m} \text{Var}(\hat{Y}_{1,m,n} + \cdots + \hat{Y}_{m,m,n}) = \Sigma^2. \quad (5.7)$$

For  $i < k$ , we have

$$\begin{aligned} \hat{Y}_{i,m,n} \hat{Y}_{k,m,n} &= a_{0,n}^2 U_{i,0,m,n} U_{k,0,m,n} + \left( \sum_{l=1}^M a_{l,n} U_{i,l,m,n} \right) \left( \sum_{q=1}^M a_{q,n} U_{k,q,m,n} \right) \\ &\quad + a_{0,n} U_{k,0,m,n} \sum_{l=1}^M a_{l,n} U_{i,l,m,n} + a_{0,n} U_{i,0,m,n} \sum_{q=1}^M a_{q,n} U_{k,q,m,n}. \end{aligned}$$

From (3.7) and Proposition 3.2, it follows that

$$\mathbb{E} U_{i,0,m,n} U_{k,0,m,n} = 0, \quad \mathbb{E} \left( \sum_{q=1}^M U_{i,0,m,n} U_{k,q,m,n} \right) = 0,$$

and

$$\mathbb{E} \left( U_{k,0,m,n} \sum_{l=1}^M U_{i,l,m,n} \right) = 2(\mathbb{E} \xi_0^2)^2 \sum_{l=1}^M \theta_n^l \left( 1_{A_1} + 1_{A_2} \sum_{j=0}^{m-1-l} \theta_n^{2j} \right).$$

Since  $i < k$  and

$$\begin{aligned} &\left( \sum_{l=1}^M U_{i,l,m,n} \right) \left( \sum_{q=1}^M U_{k,q,m,n} \right) \\ &= \left( \sum_{l=1}^{M-1} \sum_{q=l+1}^M + \sum_{q=1}^{M-1} \sum_{l=q+1}^M \right) U_{i,l,m,n} U_{k,q,m,n} + \sum_{l=1}^M U_{i,l,m,n} U_{k,l,m,n}, \end{aligned}$$

then by (3.8) and Proposition 3.2, we have

$$\sum_{l=1}^M \mathbb{E}(U_{i,l,m,n} U_{k,l,m,n}) = \sum_{l=1}^M \theta_n^{2l} (\mathbb{E} \xi_0^2)^2 \left( 1_{A_1} + \sum_{q=0}^{m-1-2l} \theta_n^{2q} 1_{A_2} \right),$$

$$\begin{aligned} &\sum_{l=1}^{M-1} \sum_{q=l+1}^M \mathbb{E}(U_{i,l,m,n} U_{k,q,m,n}) \\ &= \sum_{l=1}^{M-1} \sum_{q=l+1}^M \left( \theta_n^{l+q} (\mathbb{E} \xi_0^2)^2 \left( 1_{A_1} + 1_{A_2} \sum_{j=0}^{m-1-(l+q)} \theta_n^{2j} \right) \right) \end{aligned}$$

and

$$\begin{aligned}
& \sum_{q=1}^{M-1} \sum_{l=q+1}^M \mathbb{E}(U_{i,l,m,n} U_{k,q,m,n}) \\
&= \sum_{q=1}^{M-1} \sum_{l=q+1}^M \theta_n^{l+q} (\mathbb{E} \xi_0^2)^2 \left( 1_{A_1} + 1_{A_2} \sum_{j=0}^{m-1-(l+q)} \theta_n^{2j} \right) \\
&+ \sum_{q=1}^{M-1} \sum_{l=q+1}^M \theta_n^{l-q} (\mathbb{E} \xi_0^2)^2 \left( 1_{E_1} + 1_{E_2} \sum_{j=0}^{m-1-(l-q)} \theta_n^{2j} \right).
\end{aligned}$$

Hence we can obtain

$$\begin{aligned}
& \mathbb{E}(\hat{Y}_{i,m,n} \hat{Y}_{k,m,n}) \\
&= 2(\mathbb{E} \xi_0^2)^2 a_{0,n} \sum_{l=1}^M a_{l,n} \theta_n^l \left( 1_{A_1} + 1_{A_2} \sum_{j=0}^{m-1-l} \theta_n^{2j} \right) \\
&+ (\mathbb{E} \xi_0^2)^2 \sum_{l=1}^M a_{l,n}^2 \theta_n^{2l} \left( 1_{A_1} + 1_{A_2} \sum_{q=0}^{m-1-2l} \theta_n^{2q} \right) \\
&+ (\mathbb{E} \xi_0^2)^2 \sum_{l=1}^{M-1} \sum_{q=l+1}^M a_{l,n} a_{q,n} \theta_n^{l+q} \left( 1_{A_1} + 1_{A_2} \sum_{j=0}^{m-1-(l+q)} \theta_n^{2j} \right) \\
&+ (\mathbb{E} \xi_0^2)^2 \sum_{q=1}^{M-1} \sum_{l=q+1}^M a_{l,n} a_{q,n} \theta_n^{l+q} \left( 1_{A_1} + 1_{A_2} \sum_{j=0}^{m-1-(l+q)} \theta_n^{2j} \right) \\
&+ (\mathbb{E} \xi_0^2)^2 \sum_{q=1}^{M-1} \sum_{l=q+1}^M a_{l,n} a_{q,n} \theta_n^{l-q} \left( 1_{E_1} + 1_{E_2} \sum_{j=0}^{m-1-(l-q)} \theta_n^{2j} \right) \\
&=: I_{1,i,k,n} + I_{2,i,k,n} + I_{3,i,k,n} + I_{4,i,k,n} + I_{5,i,k,n}.
\end{aligned} \tag{5.8}$$

Furthermore, since

$$\hat{Y}_{i,m,n}^2 = a_{0,n}^2 U_{i,0,m,n}^2 + 2a_{0,n} U_{i,0,m,n} \sum_{l=1}^M a_{l,n} U_{i,l,m,n} + \left( \sum_{l=1}^M a_{l,n} U_{i,l,m,n} \right)^2$$

and

$$\begin{aligned}
\left( \sum_{l=1}^M a_{l,n} U_{i,l,m,n} \right)^2 &= \left( \sum_{l=1}^{M-1} \sum_{q=l+1}^M + \sum_{q=1}^{M-1} \sum_{l=q+1}^M \right) a_{l,n} a_{q,n} U_{i,l,m,n} U_{i,q,m,n} \\
&+ \sum_{l=1}^M a_{l,n}^2 U_{i,l,m,n}^2,
\end{aligned}$$

then from Proposition 3.1 and Proposition 3.3, we have

$$\begin{aligned}
& \mathbb{E}(\hat{Y}_{i,m,n}^2) \\
&= a_{0,n}^2 \left( \mathbb{E}\xi_0^4 + (\mathbb{E}\xi_0^2)^2 \left[ 4 \sum_{j=1}^{m-1} \theta_n^{2j} - 1 \right] \right) \\
&+ 2a_{0,n} \sum_{l=1}^M a_{l,n} \left( \theta_n^l (\mathbb{E}\xi_0^4) - \theta_n^l (\mathbb{E}\xi_0^2)^2 + 2(\mathbb{E}\xi_0^2)^2 \theta_n^l \sum_{j=1}^{m-1-l} \theta_n^{2j} \right) \\
&+ \sum_{l=1}^{M-1} \sum_{q=l+1}^M a_{l,n} a_{q,n} \left( \theta_n^{l+q} (\mathbb{E}\xi_0^4) - 2\theta_n^{l+q} (\mathbb{E}\xi_0^2)^2 + \theta_n^{q-l} (\mathbb{E}\xi_0^2)^2 \sum_{j=0}^{m-1-(q-l)} \theta_n^{2j} \right) \\
&+ \sum_{q=1}^{M-1} \sum_{l=q+1}^M a_{l,n} a_{q,n} \left( \theta_n^{l+q} (\mathbb{E}\xi_0^4) - 2\theta_n^{l+q} (\mathbb{E}\xi_0^2)^2 + \theta_n^{l-q} (\mathbb{E}\xi_0^2)^2 \sum_{p=0}^{m-1-(l-q)} \theta_n^{2p} \right) \\
&+ \sum_{l=1}^M a_{l,n}^2 \left( \theta_n^{2l} (\mathbb{E}\xi_0^4) - 2\theta_n^{2l} (\mathbb{E}\xi_0^2)^2 + (\mathbb{E}\xi_0^2)^2 \left( 2 \sum_{j=1}^{m-1} \theta_n^{2j} + 1 \right) \right). \tag{5.9}
\end{aligned}$$

Now we prove the relations (5.6) and (5.7). From (5.8), in order to show (5.6), we only prove the following claim

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1 - \theta_n^2}{m} \sum_{k=1}^m k I_{1,1,k+1,n} \\
&= 2(\mathbb{E}\xi_0^2)^2 \lim_{n \rightarrow \infty} \frac{1 - \theta_n^2}{m} a_{0,n} \sum_{k=1}^m k \sum_{l=1}^M a_{l,n} \theta_n^l \left( 1_{A_1} + 1_{A_2} \sum_{j=0}^{m-1-l} \theta_n^{2j} \right) = 0, \tag{5.10}
\end{aligned}$$

and the proofs of other terms are similar. In fact, by the definitions of  $A_1, A_2$ , we have

$$\begin{aligned}
& \sum_{k=1}^m k \sum_{l=1}^M a_{l,n} \theta_n^l \left( 1_{A_1} + 1_{A_2} \sum_{j=0}^{m-1-l} \theta_n^{2j} \right) \\
&= \sum_{l=1}^M a_{l,n} \theta_n^l \sum_{k=1}^{M+1} k 1_{A_1} + \sum_{l=1}^M a_{l,n} \theta_n^l l \left( \sum_{j=0}^{m-1-l} \theta_n^{2j} \right)
\end{aligned}$$

which implies (5.10). Next we prove (5.7). Since

$$\text{Var}(\hat{Y}_{1,m,n} + \dots + \hat{Y}_{m,m,n}) = \sum_{k=1}^m \mathbb{E}\hat{Y}_{k,m,n}^2 + 2 \sum_{i=1}^{m-1} \sum_{k=i+1}^m \mathbb{E}\hat{Y}_{k,m,n} \hat{Y}_{i,m,n},$$

then from (5.9), we have

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1 - \theta_n^2}{m} \sum_{k=1}^m \mathbb{E}\hat{Y}_{k,m,n}^2 \\
&= \left( 4a_0^2 + 4a_0 \sum_{l=1}^M a_l + 2 \sum_{l=1}^{M-1} \sum_{q=l+1}^M a_l a_q + 2 \sum_{l=1}^M a_l^2 \right) (\mathbb{E}\xi_0^2)^2.
\end{aligned}$$

Moreover, by (5.8), we have

$$\begin{aligned}
& \frac{1 - \theta_n^2}{m} \sum_{i=1}^{m-1} \sum_{k=i+1}^m I_{1,i,k,n} \\
&= 2(\mathbb{E}\xi_0^2)^2 a_{0,n} \frac{1 - \theta_n^2}{m} \sum_{l=1}^M a_{l,n} \theta_n^l \sum_{i=1}^{m-1} \left( (l+1) + \sum_{j=0}^{m-1-l} \theta_n^{2j} \right) \\
&\rightarrow 2(\mathbb{E}\xi_0^2)^2 a_0 \sum_{l=1}^M a_l.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \frac{1 - \theta_n^2}{m} \sum_{i=1}^{m-1} \sum_{k=i+1}^m I_{2,i,k,n} \rightarrow (\mathbb{E}\xi_0^2)^2 \sum_{l=1}^M a_l^2, \\
& \frac{1 - \theta_n^2}{m} \sum_{i=1}^{m-1} \sum_{k=i+1}^m I_{3,i,k,n} \rightarrow (\mathbb{E}\xi_0^2)^2 \sum_{l=1}^{M-1} \sum_{q=l+1}^M a_l a_q, \\
& \frac{1 - \theta_n^2}{m} \sum_{i=1}^{m-1} \sum_{k=i+1}^m I_{4,i,k,n} \rightarrow (\mathbb{E}\xi_0^2)^2 \sum_{q=1}^{M-1} \sum_{l=q+1}^M a_l a_q, \\
& \frac{1 - \theta_n^2}{m} \sum_{i=1}^{m-1} \sum_{k=i+1}^m I_{5,i,k,n} \rightarrow (\mathbb{E}\xi_0^2)^2 \sum_{q=1}^{M-1} \sum_{l=q+1}^M a_l a_q,
\end{aligned}$$

so, it follows that

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1 - \theta_n^2}{m} \sum_{i=1}^{m-1} \sum_{k=i+1}^m \mathbb{E} \hat{Y}_{k,m,n} \hat{Y}_{i,m,n} \\
&= (\mathbb{E}\xi_0^2)^2 \left( 2a_0 \sum_{l=1}^M a_l + \sum_{q=1}^{M-1} \sum_{l=q+1}^M a_l a_q + \left( \sum_{q=1}^M a_q \right)^2 \right).
\end{aligned}$$

From the above discussion, we have

$$\lim_{n \rightarrow \infty} \frac{1 - \theta_n^2}{m} \text{Var}(\hat{Y}_{1,m,n} + \cdots + \hat{Y}_{m,m,n}) = 4 \left( \sum_{j=0}^M a_j \right)^2 (\mathbb{E}\xi_0^2)^2.$$

At last, we need to show (5.3). Since for any  $r > 0$ , we have

$$\begin{aligned}
& \mathbb{P} \left( \frac{(1 - \theta_n^2)^{3/2}}{b_n \sqrt{n}} \left| \sum_{l=0}^{M-1} \sum_{k=n-M+1}^{n-l} a_{l,n} Y_{k,l,n} \right| \geq r \right) \\
&\leq \sum_{l=0}^{M-1} \sum_{k=n-M+1}^{n-l} \mathbb{P} \left( \frac{(1 - \theta_n^2)^{3/2}}{b_n \sqrt{n}} |a_{l,n} Y_{k,l,n}| \geq \frac{2r}{M(M+1)} \right)
\end{aligned}$$

then from the stationarity of  $Y_{k,l,n}$  ( $k = 0, 1, \dots, n-l$ ) and the fact that for any  $l$ ,  $|a_{l,n}| < N_l$  for some  $N_l > 0$ , it is enough to show that for any  $0 \leq l \leq M$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{(1 - \theta_n^2)^{3/2}}{b_n \sqrt{n}} |Y_{k,l,n}| \geq \frac{2r}{N_l M(M+1)} \right) \rightarrow -\infty. \quad (5.11)$$



However, from the definition of  $Y_{k,l,n}$  and the fact that  $\mathbb{E}(X_{l,n}X_{0,n}) = \theta_n^l(1 - \theta_n^2)^{-1}\mathbb{E}\xi_0^2$ , we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{(1 - \theta_n^2)^{3/2}}{b_n \sqrt{n}} |Y_{k,l,n}| \geq \frac{2r}{N_l M(M+1)} \right) \\ & \leq \lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{(1 - \theta_n^2)^{3/2}}{b_n \sqrt{n}} |X_{l,n}X_{0,n} - \mathbb{E}(X_{l,n}X_{0,n})| \geq \frac{2r}{N_l M(M+1)} \right) \\ & \leq \lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{(1 - \theta_n^2)^{3/2}}{b_n \sqrt{n}} |X_{l,n}X_{0,n}| \geq \frac{r}{N_l M(M+1)} \right) \\ & \leq \lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{(1 - \theta_n^2)^{1/2}}{b_n \sqrt{n}} |X_{l,n}X_{0,n}| \geq \frac{r}{N_l M(M+1)} \right) \rightarrow -\infty. \end{aligned}$$

Here the last limit is due to the similar proof in Lemma 3.2 .

## 6. PROOF OF PROPOSITION 2.1

The proof of Proposition 2.1 stems from the method of Theorem 2.2. From the definition of  $\hat{\theta}_n$ , we have

$$\begin{aligned} & \frac{\sqrt{n}}{b_n(1 - \theta_n^2)^{1/2}} (\hat{\theta}_n - \theta_n) \\ &= \frac{\sqrt{n}}{b_n(1 - \theta_n^2)^{1/2}} \frac{\sum_{k=1}^n (X_{k,n}X_{k-1,n} - \theta_n X_{k-1,n}^2)}{\sum_{k=1}^n X_{k-1,n}^2} \\ &= \frac{\frac{(1 - \theta_n^2)^{3/2}}{\sqrt{n}b_n} \sum_{k=1}^n (1 - \theta_n^2)^{-1} (\mathbb{E}\xi_0^2)^{-1} (X_{k,n}X_{k-1,n} - \theta_n X_{k-1,n}^2)}{(1 - \theta_n^2)(\mathbb{E}\xi_0^2)^{-1} \frac{1}{n} \sum_{k=1}^n X_{k-1,n}^2} =: \frac{r_n}{R_n}. \end{aligned}$$

Let us first prove that  $(R_n - 1)$  is negligible with respect to the moderate deviation principle, i.e., to show that for any  $r > 0$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( (1 - \theta_n^2)(\mathbb{E}\xi_0^2)^{-1} \frac{1}{n} \left| \sum_{k=1}^n (X_{k-1,n}^2 - \mathbb{E}X_{k-1,n}^2) \right| > r \right) = -\infty, \quad (6.1)$$

where we use the fact  $\mathbb{E}X_{k-1,n}^2 = \mathbb{E}X_{0,n}^2 = (1 - \theta_n^2)^{-1}\mathbb{E}\xi_0^2$ . Since

$$\begin{aligned} & \mathbb{P} \left( (1 - \theta_n^2)(\mathbb{E}\xi_0^2)^{-1} \frac{1}{n} \left| \sum_{k=1}^n (X_{k-1,n}^2 - \mathbb{E}X_{k-1,n}^2) \right| > r \right) \\ &= \mathbb{P} \left( \frac{(1 - \theta_n^2)^{3/2}}{b_n \sqrt{n}} \left| \sum_{k=1}^n (X_{k-1,n}^2 - \mathbb{E}X_{k-1,n}^2) \right| > \frac{r \sqrt{n}(\mathbb{E}\xi_0^2)(1 - \theta_n^2)^{1/2}}{b_n} \right), \end{aligned}$$

then by the condition (3) (which implies  $\sqrt{n(1 - \theta_n^2)}b_n^{-1} \rightarrow \infty$ ) and Theorem 2.1 yields (6.1). Next we only need to prove that  $r_n$  satisfies the moderate deviation principle. Let

$$a_{0,n} = -(1 - \theta_n^2)^{-1}(\mathbb{E}\xi_0^2)^{-1}\theta_n \quad \text{and} \quad a_{1,n} = (1 - \theta_n^2)^{-1}(\mathbb{E}\xi_0^2)^{-1},$$

then

$$r_n = \frac{(1 - \theta_n^2)^{3/2}}{\sqrt{n}b_n} \sum_{l=0}^1 \sum_{k=0}^{n-1} a_{l,n} (X_{k+l,n}X_{k,n} - \mathbb{E}X_{k+l,n}X_{k,n}).$$

Since  $a_{0,n} \rightarrow -\infty, a_{1,n} \rightarrow \infty$ , then we can not use directly Theorem 2.2 to prove the moderate deviation of  $r_n$ . So we need to slightly modify the proof of Theorem 2.2.

Now rewrite  $r_n$  as

$$\begin{aligned} r_n &= \frac{(1 - \theta_n^2)^{1/2}}{\sqrt{n}b_n(\mathbb{E}\xi_0^2)} \sum_{k=1}^n \xi_k \sum_{p=0}^{\infty} \theta_n^p \xi_{k-1-p} \\ &=: \frac{1}{\sqrt{n}b_n} \sum_{k=1}^n \hat{X}_{k,n}. \end{aligned}$$

Let  $m$  be a increasing sequence satisfying the properties in (4.4) and put

$$\hat{X}_{k,m,n} = \frac{(1 - \theta_n^2)^{1/2}}{(\mathbb{E}\xi_0^2)} \sum_{p=0}^{m-1} \theta_n^p \xi_{k-1-p} \xi_k$$

then  $\{\hat{X}_{k,m,n}\}$  is a strictly stationary sequence with  $m$ -dependent structure.

**Lemma 6.1.** *For any  $r > 0$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{1}{b_n \sqrt{n}} \left| \sum_{k=1}^n \hat{X}_{k,m,n} \right| \geq r \right) = -\frac{r^2}{2}.$$

*Proof.* In order to obtain the desired result, it is enough to check the conditions in Lemma 3.3. Firstly, it is easy to see that

$$\lim_{n \rightarrow \infty} \mathbb{E} \hat{X}_{k,m,n}^2 = 1$$

and for any  $k \neq j$ ,

$$\mathbb{E}(\hat{X}_{k,m,n} \hat{X}_{j,m,n}) = 0.$$

Hence we have

$$\lim_{n \rightarrow \infty} \frac{1}{m} \text{Var}(\hat{X}_{1,m,n} + \cdots + \hat{X}_{m,m,n}) = 1$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m i \mathbb{E}(\hat{X}_{1,m,n} \hat{X}_{i+1,m,n}) = 0.$$

Moreover, by using the similar proofs of (4.8) and (4.9), the conditions (B) and (C) in Lemma 3.3 hold. So we complete the proof by using Lemma 3.3.  $\square$

**Lemma 6.2.** *For any  $r > 0$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{1}{b_n \sqrt{n}} \left| \sum_{k=1}^n (\hat{X}_{k,m,n} - \hat{X}_{k,n}) \right| \geq r \right) = -\infty.$$

*Proof.* Since for any  $0 \leq j \leq n$ ,

$$\begin{aligned} \sum_{k=1}^j (\hat{X}_{k,m,n} - \hat{X}_{k,n}) &= \frac{(1 - \theta_n^2)^{1/2}}{(\mathbb{E}\xi_0^2)} \sum_{k=1}^j \sum_{p=m}^{\infty} \theta_n^p \xi_{k-1-p} \xi_k \\ &= \frac{(1 - \theta_n^2)^{1/2}}{(\mathbb{E}\xi_0^2)} \theta_n^m \sum_{k=1}^j \sum_{p=0}^{\infty} \theta_n^p \xi_{k-1-p-m} \xi_k \\ &= \frac{(1 - \theta_n^2)^{1/2}}{(\mathbb{E}\xi_0^2)} \theta_n^m \sum_{p=0}^{\infty} \theta_n^p \sum_{k=1}^j W_{k,m+p+1} \end{aligned}$$

where  $W_{k,m+p+1}$  is defined in (4.12), then by the same proof of Lemma 4.1, the desired result can be obtained.  $\square$

At last, Proposition 2.1 can be given by using Lemma 6.1 and 6.2.

## 7. APPENDIX

**Proof of Lemma 3.3.** For each  $n$ , let

$$Y_{j,n} = \sum_{i=1}^m X_{(j-1)m+i,n}, \quad 1 \leq j \leq l$$

where  $l := l(n) = \max\{j : jm \leq n\}$ , then  $\{Y_{1,n}, \dots, Y_{l,n}\}$  are 1-dependent random variables. Furthermore, take  $p = p(n)$ , such that

$$p(n) \rightarrow \infty \quad \text{and} \quad \frac{b_n(mp)^{1+\gamma}}{\sqrt{n}} \rightarrow 0, \quad \text{as } n \rightarrow \infty \quad (7.1)$$

and define

$$Z_{h,n} = \sum_{(h-1)p < j < hp} Y_{j,n}, \quad 1 \leq h \leq t$$

where  $t := t(n) = \max\{h, hp < l\}$ , then  $\{Z_{1,n}, \dots, Z_{t,n}\}$  is an i.i.d. random sequence, and we have the following relations

$$\begin{aligned} \sum_{i=1}^n X_{i,n} &= \sum_{j=1}^l Y_{j,n} + \sum_{i=lm+1}^n X_{i,n} \\ &= \sum_{h=1}^t Z_{h,n} + \sum_{j=tp+1}^l Y_{j,n} + \sum_{h=1}^t Y_{hp,n} + \sum_{i=lm+1}^n X_{i,n}. \end{aligned} \quad (7.2)$$

**Lemma 7.1.** *Under the assumptions of Lemma 3.3, for any  $\lambda \in \mathbb{R}$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{E} \exp \left( \lambda \frac{b_n}{\sqrt{n}} \sum_{h=1}^t Z_{h,n} \right) = \frac{\lambda^2 \sigma^2}{2},$$

i.e., for any  $r > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{1}{b_n \sqrt{n}} \left| \sum_{h=1}^t Z_{h,n} \right| > r \right) = -\frac{r^2}{2\sigma^2}.$$

*Proof.* For  $\tau > 0$ , define

$$X_{i,n}^\tau := X_{i,n} \mathbb{I}_{\{|X_{i,n}| \leq \tau \frac{\sqrt{n}}{b_n}\}}, \quad 1 \leq i \leq n,$$

$$Y_{j,n}^\tau = \sum_{i=1}^m X_{(j-1)m+i,n}^\tau, \quad 1 \leq j \leq l$$

and

$$Z_{h,n}^\tau = \sum_{(h-1)p < j < hp} Y_{j,n}^\tau, \quad 1 \leq h \leq t$$

where  $l, p, t$  are defined in the above notations. Now we divide the proof into the following two steps.

**Step 1.** We claim that for any  $r > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{1}{b_n \sqrt{n}} \left| \sum_{h=1}^t Z_{h,n}^\tau \right| > r \right) = -\frac{r^2}{2\sigma^2}. \quad (7.3)$$

Since  $\{Z_{1,n}^\tau, \dots, Z_{t,n}^\tau\}$  is an i.i.d. random sequence, then for  $\lambda \in \mathbb{R}$ ,

$$\begin{aligned} & \frac{1}{b_n^2} \log \mathbb{E} \exp \left( \lambda \frac{b_n}{\sqrt{n}} \sum_{h=1}^t Z_{h,n}^\tau \right) = \frac{t}{b_n^2} \log \mathbb{E} \exp \left( \lambda \frac{b_n}{\sqrt{n}} Z_{1,n}^\tau \right) \\ &= \frac{t}{b_n^2} \log \left( 1 + \lambda \frac{b_n}{\sqrt{n}} \mathbb{E} Z_{1,n}^\tau + \frac{\lambda^2 b_n^2}{2n} \mathbb{E} (Z_{1,n}^\tau)^2 + O \left( \frac{\lambda^3 b_n^3}{n^{3/2}} \mathbb{E} (Z_{1,n}^\tau)^3 \right) \right). \end{aligned}$$

First from the conditions (A), (B), (C) and Fubini Theorem, we have

$$\begin{aligned} & m \mathbb{E} \left( X_{1,n}^2 \mathbb{I}_{\{|X_{1,n}| > \tau \frac{\sqrt{n}}{b_n}\}} \right) \\ &= \frac{n}{b_n^2 m} \mathbb{E} \left( \left( \frac{b_n m}{\sqrt{n}} X_{1,n} \right)^2 \mathbb{I}_{\{|X_{1,n}| > \tau \frac{\sqrt{n}}{b_n}\}} \right) \\ &= \frac{2n}{b_n^2 m} \int_0^\infty x \mathbb{P} \left( |X_{1,n}| \geq \frac{\sqrt{n}x}{b_n m}, |X_{1,n}| > \tau \frac{\sqrt{n}}{b_n} \right) dx \\ &= \frac{2n}{b_n^2 m} \int_{\tau m}^\infty x \mathbb{P} \left( |X_{1,n}| \geq \frac{\sqrt{n}x}{b_n m} \right) dx + \frac{2n}{b_n^2 m} \int_0^{\tau m} x \mathbb{P} \left( |X_{1,n}| > \tau \frac{\sqrt{n}}{b_n} \right) dx \quad (7.4) \\ &= \frac{2n}{b_n^2 m} \int_{\tau m}^\infty x \mathbb{P} \left( |X_{1,n}| \geq \frac{\sqrt{n}x}{b_n m} \right) dx + \frac{n\tau^2 m}{b_n^2} \mathbb{P} \left( |X_{1,n}| > \tau \frac{\sqrt{n}}{b_n} \right) \\ &\leq \frac{2n}{b_n^2 m} \int_{\tau m}^\infty x \mathbb{P} \left( |X_{1,n}| \geq \frac{\sqrt{n}x}{b_n m} \right) dx + \tau^2 \left( \frac{\sqrt{n}}{b_n} \right)^{2+\frac{1}{1+\gamma}} \mathbb{P} \left( |X_{1,n}| > \tau \frac{\sqrt{n}}{b_n} \right) \\ &\rightarrow 0, \end{aligned}$$

where we utilized the fact: for all sufficiently large  $n$ ,

$$m \leq \left( \frac{\sqrt{n}}{b_n} \right)^{\frac{1}{1+\gamma}}.$$

Noting that  $\mathbb{E}X_{1,n} = 0$  and the fact  $(tmp)/n \rightarrow 1$ , we have, by (7.4),

$$\begin{aligned} \frac{t}{b_n^2} \frac{\lambda b_n}{\sqrt{n}} |\mathbb{E}Z_{1,n}^\tau| &\leq \frac{\lambda t(p-1)m}{b_n \sqrt{n}} \mathbb{E} \left( |X_{1,n}| \mathbb{I}_{\{|X_{1,n}| > \tau \frac{\sqrt{n}}{b_n}\}} \right) \\ &\leq \frac{\lambda t p m}{\tau n} \mathbb{E} \left( X_{1,n}^2 \mathbb{I}_{\{|X_{1,n}| > \tau \frac{\sqrt{n}}{b_n}\}} \right) \rightarrow 0. \end{aligned}$$

From the definition of  $Z_{1,n}^\tau$ , we have

$$\begin{aligned} \mathbb{E}(Z_{1,n}^\tau)^2 &= (p-1) \mathbb{E}(Y_{1,n}^\tau)^2 + 2 \sum_{j=1}^{p-2} \sum_{i=j+1}^{p-1} \mathbb{E}(Y_{j,n}^\tau Y_{i,n}^\tau) \\ &= (p-1) \mathbb{E}(Y_{1,n}^\tau)^2 + 2 \sum_{j=1}^{p-2} \mathbb{E}(Y_{j,n}^\tau Y_{j+1,n}^\tau) + 2 \sum_{j=1}^{p-3} \sum_{i=j+2}^{p-1} \mathbb{E}(Y_{j,n}^\tau) \mathbb{E}(Y_{i,n}^\tau). \end{aligned}$$

Since the estimate (7.4) implies

$$\begin{aligned} &\frac{1}{m} \mathbb{E}(Y_{1,n}^\tau - Y_{1,n})^2 \\ &= \frac{1}{m} \mathbb{E} \left( X_{1,n} \mathbb{I}_{\{|X_{1,n}| > \tau \frac{\sqrt{n}}{b_n}\}} + \cdots + X_{m,n} \mathbb{I}_{\{|X_{m,n}| > \tau \frac{\sqrt{n}}{b_n}\}} \right)^2 \\ &\leq m \mathbb{E} \left( X_{1,n}^2 \mathbb{I}_{\{|X_{1,n}| > \tau \frac{\sqrt{n}}{b_n}\}} \right) \rightarrow 0 \end{aligned}$$

then we have

$$\frac{1}{m} \mathbb{E}(Y_{1,n}^\tau)^2 \rightarrow \sigma^2$$

where we used the condition (D) and the triangle inequality in the  $L^2$  spaces:

$$\|Y_{1,n}^\tau\|_2 \leq \|Y_{1,n}^\tau - Y_{1,n}\|_2 + \|Y_{1,n}\|_2, \quad \|Y_{1,n}\|_2 \leq \|Y_{1,n}^\tau - Y_{1,n}\|_2 + \|Y_{1,n}^\tau\|_2.$$

So we have

$$\frac{\lambda^2 t}{n} (p-1) \mathbb{E}(Y_{1,n}^\tau)^2 = \frac{\lambda^2 t m (p-1)}{n} \frac{\mathbb{E}(Y_{1,n}^\tau)^2}{m} \rightarrow \lambda^2 \sigma^2.$$

Similarly, we have

$$\begin{aligned} &\frac{1}{m} \sum_{i=1}^m i \mathbb{E} |(X_{1,n}^\tau - X_{1,n})(X_{i+1,n}^\tau - X_{i+1,n})| \\ &\leq \frac{m+1}{2} \mathbb{E} \left( X_{1,n}^2 \mathbb{I}_{\{|X_{1,n}| > \tau \frac{\sqrt{n}}{b_n}\}} \right) \rightarrow 0, \end{aligned} \tag{7.5}$$

$$\begin{aligned} &\frac{1}{m} \sum_{i=1}^m i \mathbb{E} |(X_{1,n}^\tau - X_{1,n})X_{i+1,n}| \\ &= \frac{1}{m} \sum_{i=1}^m i \left\{ \mathbb{E} \left( |X_{1,n}| \mathbb{I}_{\{|X_{1,n}| > \tau \frac{\sqrt{n}}{b_n}\}} |X_{i+1,n}| \mathbb{I}_{\{|X_{i+1,n}| \leq \tau \frac{\sqrt{n}}{b_n}\}} \right) \right. \\ &\quad \left. + \mathbb{E} \left( |X_{1,n}| \mathbb{I}_{\{|X_{1,n}| > \tau \frac{\sqrt{n}}{b_n}\}} |X_{i+1,n}| \mathbb{I}_{\{|X_{i+1,n}| > \tau \frac{\sqrt{n}}{b_n}\}} \right) \right\} \\ &\leq (m+1) \mathbb{E} \left( X_{1,n}^2 \mathbb{I}_{\{|X_{1,n}| > \tau \frac{\sqrt{n}}{b_n}\}} \right) \rightarrow 0, \end{aligned} \tag{7.6}$$

and

$$\frac{1}{m} \sum_{i=1}^m i \mathbb{E} |(X_{i+1,n}^\tau - X_{i+1,n}) X_{1,n}| \rightarrow 0, \quad (7.7)$$

from which we can deduce

$$\begin{aligned} & \frac{1}{m} \sum_{i=1}^m i \mathbb{E} (X_{1,n}^\tau X_{i+1,n}^\tau) \\ &= \frac{1}{m} \sum_{i=1}^m i [\mathbb{E} (X_{1,n} X_{i+1,n}) + \mathbb{E} (X_{1,n}^\tau - X_{1,n}) (X_{i+1,n}^\tau - X_{i+1,n}) \\ & \quad + \mathbb{E} (X_{1,n}^\tau - X_{1,n}) X_{i+1,n} + \mathbb{E} X_{1,n} (X_{i+1,n}^\tau - X_{i+1,n})] \rightarrow 0. \end{aligned}$$

Hence we can get

$$\begin{aligned} & \frac{t\lambda^2}{n} \sum_{j=1}^{p-2} \mathbb{E} (Y_{j,n}^\tau Y_{j+1,n}^\tau) \\ &= \frac{\lambda^2 t(p-2)m}{n} \left\{ \frac{1}{m} \sum_{i=1}^m i \mathbb{E} (X_{1,n}^\tau X_{i+1,n}^\tau) + \frac{1}{m} \frac{m(m-1)}{2} (\mathbb{E} X_{1,n}^\tau)^2 \right\} \rightarrow 0 \end{aligned}$$

and

$$\frac{t\lambda^2}{n} \sum_{j=1}^{p-3} \sum_{i=j+2}^{p-1} |\mathbb{E} (Y_{j,n}^\tau) \mathbb{E} (Y_{i,n}^\tau)| \leq \frac{\lambda^2 t p^2 m^2}{n} (\mathbb{E} X_{1,n}^\tau)^2 \rightarrow 0$$

where we used the condition (7.1) and the fact that

$$(\mathbb{E} X_{1,n}^\tau)^2 = \left( \mathbb{E} X_{1,n} \mathbb{I}_{\{|X_{1,n}| > \tau \frac{\sqrt{n}}{b_n}\}} \right)^2 \leq \frac{b_n^2}{n\tau^2} \left( \mathbb{E} X_{1,n}^2 \mathbb{I}_{\{|X_{1,n}| > \tau \frac{\sqrt{n}}{b_n}\}} \right)^2.$$

So we have

$$\frac{\lambda^2 t}{n} \mathbb{E} (Z_{1,n}^\tau)^2 \rightarrow \lambda^2 \sigma^2. \quad (7.8)$$

Furthermore, for any  $\varepsilon > 0$ , we have for all  $n$  sufficient large,

$$\begin{aligned} \frac{\lambda^3 b_n t}{n^{3/2}} \mathbb{E} |Z_{1,n}^\tau|^3 &\leq \frac{\lambda^3 b_n t}{n^{3/2}} \mathbb{E} \left( |Z_{1,n}^\tau|^2 \left( \sum_{i=1}^{(p-1)m} |X_{i,n}^\tau| \mathbb{I}_{\{|X_{i,n}^\tau| \leq \varepsilon \frac{\sqrt{n}}{b_n m p}\}} \right) \right) \\ &\quad + \frac{\lambda^3 b_n t}{n^{3/2}} \mathbb{E} \left( |Z_{1,n}^\tau|^2 \left( \sum_{i=1}^{(p-1)m} |X_{i,n}^\tau| \mathbb{I}_{\{|X_{i,n}^\tau| > \varepsilon \frac{\sqrt{n}}{b_n m p}\}} \right) \right) \\ &\leq \frac{\varepsilon \lambda^3 t}{n} \mathbb{E} |Z_{1,n}^\tau|^2 + \frac{\tau^3 \lambda^3 t m^3 p^3}{b_n^2} \mathbb{P} \left\{ |X_{1,n}| > \varepsilon \frac{\sqrt{n}}{b_n m p} \right\} \\ &= \frac{\varepsilon \lambda^3 t}{n} \mathbb{E} |Z_{1,n}^\tau|^2 + \frac{\tau^3 \lambda^3 t m p}{n} \frac{n m^2 p^2}{b_n^2} \mathbb{P} \left\{ |X_{1,n}| > \varepsilon \frac{\sqrt{n}}{b_n m p} \right\} \\ &\leq \frac{\varepsilon \lambda^3 t}{n} \mathbb{E} |Z_{1,n}^\tau|^2 + \frac{\tau^3 \lambda^3 t m p}{n} \left( \frac{\sqrt{n}}{b_n} \right)^{2+\frac{2}{1+\gamma}} \mathbb{P} \left\{ |X_{1,n}| > \varepsilon \left( \frac{\sqrt{n}}{b_n} \right)^{1-\frac{1}{1+\gamma}} \right\}, \end{aligned}$$

where we used the condition

$$\lim_{n \rightarrow \infty} \frac{b_n (m p)^{1+\gamma}}{\sqrt{n}} = 0 \implies \lim_{n \rightarrow \infty} \frac{m p}{(\sqrt{n}/b_n)^{1/(1+\gamma)}} = 0.$$

Therefore, by noting  $mpt/n \rightarrow 1$  and the arbitrariness of  $\varepsilon$ , we have, from the relation (7.8) and the condition (C),

$$\frac{\lambda^3 b_n t}{n^{3/2}} \mathbb{E} |Z_{1,n}^\tau|^3 \rightarrow 0.$$

Hence from the above discussions, we have

$$\frac{1}{b_n^2} \log \mathbb{E} \exp \left( \lambda \frac{b_n}{\sqrt{n}} \sum_{h=1}^t Z_{h,n}^\tau \right) \rightarrow \frac{1}{2} \lambda^2 \sigma^2,$$

which yields, by the Gärtner-Ellis theorem, for any  $r > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{1}{b_n \sqrt{n}} \left| \sum_{h=1}^t Z_{h,n}^\tau \right| > r \right) = -\frac{r^2}{2\sigma^2}.$$

**Step 2.** We shall prove that for any  $r > 0$  and  $\tau > 0$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{1}{b_n \sqrt{n}} \left| \sum_{h=1}^t (Z_{h,n}^\tau - Z_{h,n}) \right| \geq r \right) = -\infty. \quad (7.9)$$

Let  $[a]$  denote the integral part of  $a$ , then for any  $\lambda > 0$ ,

$$\begin{aligned} & \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{1}{b_n \sqrt{n}} \left| \sum_{h=1}^t (Z_{h,n}^\tau - Z_{h,n}) \right| \geq r \right) \\ & \leq -r\lambda + \frac{1}{b_n^2} \log \mathbb{E} \exp \left( \frac{\lambda b_n}{\sqrt{n}} \sum_{h=1}^t |Z_{h,n}^\tau - Z_{h,n}| \right) \\ & \leq -r\lambda + \frac{t}{b_n^2} \log \mathbb{E} \exp \left( \frac{\lambda b_n}{\sqrt{n}} \sum_{j=1}^{p-1} |Y_{j,n}^\tau - Y_{j,n}| \right) \\ & \leq -r\lambda + \frac{t}{b_n^2} \log \left( \mathbb{E} \exp \left( 2 \frac{\lambda b_n}{\sqrt{n}} \sum_{j=1}^{[(p-1)/2]} |Y_{2j,n}^\tau - Y_{2j,n}| \right) \right)^{1/2} \\ & \quad \times \left( \mathbb{E} \exp \left( 2 \frac{\lambda b_n}{\sqrt{n}} \sum_{j=1}^L |Y_{2j-1,n}^\tau - Y_{2j-1,n}| \right) \right)^{1/2} \\ & = -r\lambda + \frac{t(p-1)}{2b_n^2} \log \mathbb{E} \exp \left( 2 \frac{\lambda b_n}{\sqrt{n}} |Y_{1,n}^\tau - Y_{1,n}| \right) \end{aligned} \quad (7.10)$$

where

$$L := L(n) := \begin{cases} [(p-1)/2] & \text{if } p-1 \text{ is even} \\ [(p-1)/2] + 1 & \text{if } p-1 \text{ is odd.} \end{cases}$$

By the definitions of  $Y_{1,n}^\tau$ ,  $Y_{1,n}$  and the elementary inequality  $\log x \leq x - 1$  for all  $x > 0$ , then we have

$$\begin{aligned}
& \frac{t(p-1)}{2b_n^2} \log \mathbb{E} \exp \left( 2 \frac{\lambda b_n}{\sqrt{n}} |Y_{1,n}^\tau - Y_{1,n}| \right) \\
& \leq \frac{tp}{2b_n^2} \left( \mathbb{E} \exp \left( 2 \frac{\lambda b_n m}{\sqrt{n}} |X_{1,n}^\tau - X_{1,n}| \right) - 1 \right) \\
& \leq \frac{tp}{2b_n^2} (e^M - 1) \mathbb{P} \left( |X_{1,n}| > \tau \frac{\sqrt{n}}{b_n} \right) + \frac{tp}{2b_n^2} \int_M^\infty e^x \mathbb{P} \left( |X_{1,n}| \geq \frac{\sqrt{nx}}{2\lambda b_n m} \right) dx \\
& \leq \frac{tp}{2n\tau^2} (e^M - 1) \mathbb{E} \left( X_{1,n}^2 \mathbb{I}_{\{|X_{1,n}| > \tau \frac{\sqrt{n}}{b_n}\}} \right) + \frac{n}{2b_n^2 m} \int_M^\infty e^x \mathbb{P} \left( |X_{1,n}| \geq \frac{\sqrt{nx}}{2\lambda b_n m} \right) dx \\
& \rightarrow 0
\end{aligned}$$

where we used (7.4) and the condition (B). Thus the desired result follows from (7.10) and the arbitrariness of  $\lambda$ .  $\square$

**Lemma 7.2.** *Under the assumptions of Lemma 3.3, for any  $r > 0$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( \frac{1}{b_n \sqrt{n}} \left| \sum_{j=tp+1}^l Y_{j,n} + \sum_{h=1}^t Y_{hp,n} + \sum_{i=lm+1}^n X_{i,n} \right| > r \right) = -\infty.$$

*Proof.* Taking along the lines of the proof of Lemma 7.1, we can prove that anyone of the following three terms

$$\sum_{j=tp+1}^l Y_{j,n}, \quad \sum_{h=1}^t Y_{hp,n} \quad \text{and} \quad \sum_{i=lm+1}^n X_{i,n}$$

can be negligible with respect to the moderate deviation principle.  $\square$

At last, based on Lemma 7.1 and Lemma 7.2, the proof of Lemma 3.3 can be finished.  $\square$

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(Y. Miao) COLLEGE OF MATHEMATICS AND INFORMATION SCIENCE, HENAN NORMAL UNIVERSITY, HENAN PROVINCE, 453007, CHINA.

*E-mail address:* yumiao728@yahoo.com.cn

(Y. -L. Wang) COLLEGE OF MATHEMATICS AND INFORMATION SCIENCE, HENAN NORMAL UNIVERSITY, HENAN PROVINCE, 453007, CHINA.

(G. -Y. Yang) DEPARTMENT OF MATHEMATICS, ZHENGZHOU UNIVERSITY, HENAN PROVINCE, 450001, CHINA.

*E-mail address:* study\_yang@yahoo.com.cn